MATHEMATICS CLASS

from October 22 to November 5, 2020

Exercise 1. Solve the following inequalities, which can be be transformed in suitable second-degree inequalities.

$$a.\ -9x^2 + 12x - 4 \geq 0 \qquad\qquad b.\ x^4 - 2x^2 - 8 \leq 0$$

Solutions

$$a.\ \mathcal{S} = \{\tfrac{2}{3}\} \qquad\qquad b.\ \mathcal{S} = [-2, 2]$$

Exercise 2. Solve the following inequalities.

$$a.\ (x^2 + 2x - 8)(x + 1) > 0 \qquad\qquad b.\ \frac{x^2 - x - 2}{x^2 - 3x} \leq 0$$

Solutions

$$a.\ \mathcal{S} =]-4, -1[\, \cup\,]2, +\infty[\qquad\qquad b.\ \mathcal{S} = [-1, 0[\, \cup\, [2, 3[$$

Exercise 3. Solve the following inequalities.

$$a.\ \sqrt{x^2 - 4x + 2} > x + 3 \qquad\qquad b.\ \sqrt{1 + x} + \sqrt{1 - x} > 1$$

Solutions

$$c.\ \mathcal{S} =]-\infty, -\tfrac{7}{10}[\qquad\qquad d.\ \mathcal{S} = [-1, 1]$$

Exercise 4. Solve the following inequalities either with the absolute value or with the square root.

$$a.\ -1 \leq x + |x - 2| < 3 \qquad b.\ |x + 3| > |x - 2| \qquad c.\ \sqrt{3x - 1} \leq |x + 1|$$

Solutions

$$a.\ \mathcal{S} = \left]-\infty, \tfrac{5}{2}\right[\qquad b.\ \mathcal{S} =]-\tfrac{1}{2}, +\infty[\qquad c.\ \mathcal{S} = \left[\tfrac{1}{3}, +\infty\right[$$

Exercise 5. Determine the domain of the following functions:

$$f(x) = \sqrt{x + 1} + \sqrt{3 - 2x}, \qquad g(x) = \frac{(x + 5)(x - 1)}{(x^2 + 1)(x^2 - 3)}, \qquad h(x) = \frac{1 + \sqrt{x}}{x^3(1 - \sqrt{3x})}$$

Solutions

$$\operatorname{dom}(f) = \left\{x \in \mathbb{R} : -1 \le x \le \frac{3}{2}\right\}, \qquad \operatorname{dom}(g) = \{x \in \mathbb{R} : x \ne \mp\sqrt{3}\},$$

$$\operatorname{dom}(h) = \left\{x \in \mathbb{R} : (0 < x < \frac{1}{3}) \text{ or } (x > \frac{1}{3})\right\}$$

Exercise 6. Determine the domain of the following function and compute the image of the set A:

$$h(x) = \frac{\sqrt{x-3}}{x-4}, \qquad A = \{3, 7, 9, 12\}.$$

Solution

$$\operatorname{dom}(h) = \{x \in \mathbb{R} : (3 \le x < 4) \text{ or } (x > 4)\}, \qquad h(A) = \left\{0, \frac{3}{8}, \frac{\sqrt{6}}{5}, \frac{2}{3}\right\}.$$

Exercise 7. Determine the domain of the following function and compute the preimage of the set B:

$$h(x) = \sqrt{x-3}, \quad B = \{-1, 1, 2, 3\}.$$

Solution

$$\operatorname{dom}(h) = \{x \in \mathbb{R} : x \ge 3\}, \qquad h^{-1}(B) = \{4, 7, 12\}.$$

Exercise 8. For the following couples of functions $f, g : \mathbb{R} \to \mathbb{R}$, determine the expression of the composite functions $f \circ g$ and $g \circ f$.

$$i) \quad f(x) = x^2 - 1, \quad g(x) = \frac{1}{x^2+1}$$

$$ii) \quad f(x) = \sqrt{\frac{x+1}{2x^2+x+1}}, \quad g(x) = 3x^3$$

Solutions

$$i) \quad (f \circ g)(x) = \frac{1-(x^2+1)^2}{(x^2+1)^2}, \quad (g \circ f)(x) = \frac{1}{(x^2-1)^2+1};$$

$$ii) \quad (f \circ g)(x) = \sqrt{\frac{3x^3+1}{18x^6+3x^3+1}}, \quad (g \circ f)(x) = \frac{3(x+1)}{2x^2+x+1}\sqrt{\frac{x+1}{2x^2+x+1}}$$

Exercise 9. Determine the image set of each function, then draw its graph and determine its inverse function.

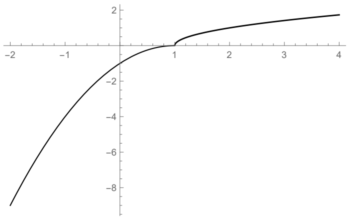
$$f(x) = \begin{cases} -(x-1)^2 & \text{if } x < 1 \\ \sqrt{x-1} & \text{if } x \ge 1 \end{cases}, \qquad h(x) = \begin{cases} x^3 - 2 & \text{if } x < 0 \\ \sqrt{x} + 2 & \text{if } x \ge 0 \end{cases}$$

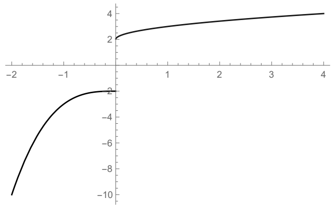
Solutions

$$f^{-1}:\mathbb{R}\to\mathbb{R} \qquad\qquad f^{-1}(y)=\begin{cases}1-\sqrt{-y} & \text{if } y<0\\ y^2+1 & \text{if } y\geq 0\end{cases}$$

$$h^{-1}:\,]-\infty,-2[\,\cup\,[2,+\infty[\,\to\mathbb{R} \qquad\qquad h^{-1}(y)=\begin{cases}\sqrt[3]{y+2} & \text{if } y<-2\\ (y-2)^2 & \text{if } y\geq 2\end{cases}$$

The graphs of the functions f, h are, respectively,





Exercise 10. Determine the inverse function $f^{-1}(y)$ of the function

$$f(x)=\begin{cases}-x^2 & \text{if } x\leq 0\\ \frac{1}{x^3} & \text{if } x>0\end{cases}$$

Solution

$$f^{-1}:\mathbb{R}\to\mathbb{R}, \quad f^{-1}(y)=\begin{cases}-\sqrt{-y} & \text{if } y\leq 0\\ \sqrt[3]{\frac{1}{y}} & \text{if } y>0\end{cases}$$

Exercise 11. Say which of the following are bijective functions and, for each of them, find the inverse function.

$$f:\mathbb{R}\to\mathbb{R}, \quad f(x)=2x+3,$$
$$g:\mathbb{R}\to\mathbb{R}, \quad g(x)=x^2-6x+9$$

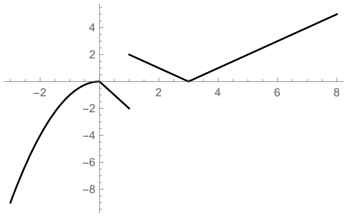
Solution
f is a bijective function, $f^{-1}:\mathbb{R}\to\mathbb{R}$, $f^{-1}(y)=\frac{y-3}{2}$. g is not a bijective function, since it is neither injective nor surjective.

Exercise 12. Consider the following piecewise function

$$f:\mathbb{R}\to\mathbb{R}, \quad f(x)=\begin{cases}-x^2 & \text{if } x<0,\\ -2x & \text{if } 0\leq x<1,\\ |x-3| & \text{if } x\geq 1.\end{cases}$$

(i) Draw the graph of the function, then determine the image of the three intervals in which f is defined.
(ii) Compute $f(-1)$, $f(0)$, $f(1)$, $f(5)$, $f^{-1}(-\frac{1}{6})$, $f^{-1}(3)$.
(iii) Is the function surjective?
(iv) Is the function injective?

Solution (i)–(iii)



The images through f of the sets above-mentioned are: $f(]-\infty,0[)=]\infty,0[$, $f([0,1[)=]-2,0]$, $f([1,+\infty[)=[0,+\infty[$, then $f(\mathbb{R})=f(]-\infty,0[)\cup f([0,1[)\cup([1,+\infty[)=\mathbb{R}$. As a consequence the function $f:\mathbb{R}\to\mathbb{R}$ is surjective.
(ii) $f(-1)=-1$, $f(0)=0$, $f(1)=f(5)=2$, $f^{-1}(-\frac{1}{6})=\{-\frac{1}{\sqrt{6}},\frac{1}{12}\}$, $f^{-1}(3)=6$.
(iv) The function f is not injective, indeed, for all $y\in]-2,2]$, there exist $x_1,x_2\in\mathbb{R}$ with $x_1\neq x_2$, such that $f(x_1)=y=f(x_2)$.

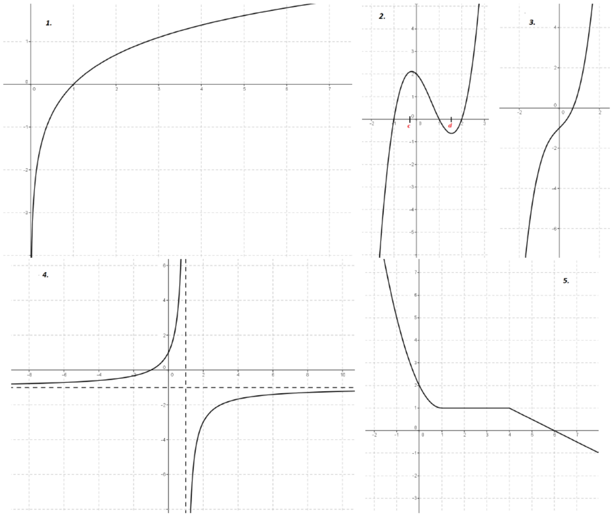
Exercise 13. For each of the following sets, say if it is lower bounded and upper bounded, find its infimum and its supremum, and establish if they are its minimum and its maximum respectively.

$$A=\{5-n^3 : n\in\mathbb{N}\},\qquad B=\left\{\frac{n}{n+1} : n\in\mathbb{N}\right\},$$

$$C=\left\{\frac{2n}{n^2+1} : n\in\mathbb{N}\right\},\qquad D=\{x\in\mathbb{R} : x^2\leq 11\}.$$

Solutions
A is lower unbounded, that is, $\inf(A)=-\infty$. On the other hand A is upper bounded and $\sup(A)=\max(A)=5$.
B : we notice that $0\leq\frac{n}{n+1}<1$ for every $n\in\mathbb{N}$, hence B is bounded. In particular $\inf(B)=\min(B)=0$ and $\sup(B)=1$.
C : we notice that $0\leq\frac{2n}{n^2+1}\leq 1$ for every $n\in\mathbb{N}$, hence C is bounded. In particular $\inf(C)=\min(C)=0$ and $\sup(C)=\max(C)=1$.
D is bounded with $\inf(D)=\min(D)=-\sqrt{11}$ and $\sup(D)=\max(D)=\sqrt{11}$.

Exercise 14. Recognise which graphs represent monotone functions and, for each of the remaining ones, determine the maximal intervals of monotonicity.



Solutions

1. The function is increasing in its domain, which is $]0, +\infty[$, then it is strictly monotone in $]0, +\infty[$.
2. The function is increasing in $]-\infty, c]$ and in $[d, +\infty[$, decreasing in $[c, d]$ (see the graph: c is a local maximum point, d a local minimum point of the function).
3. The f unction is increasing in $\mathbb{R}$, then it is strictly monotone in $\mathbb{R}$.
4. The function is increasing in $]-\infty, 1[$ and in $]1, +\infty[$, but not in the whole domain $]-\infty, 1[\cup]1, +\infty[$.
5. The function is non-increasing in $\mathbb{R}$, then it is monotone in $\mathbb{R}$, in particular it is decreasing in $]-\infty, 1]$ and in $[4, +\infty[$, constant (non-decreasing and non-increasing) in $[1, 4]$.

Exercise 15. Determine the domain of the following functions:

1. $f(x) = \dfrac{4 + \cos x}{(\sin x)^2 - 1}$ 2. $f(x) = \arcsin(3x - x^2)$

3. $f(x) = \tan(x + 6)$ 4. $f(x) = \arctan\left(\dfrac{\sqrt[3]{x} + 1}{\sqrt{x} - 1}\right)$

5. $f(x) = \dfrac{1}{\sqrt{3}\sin x - 3\cos x}$ 6. $f(x) = \dfrac{\sqrt{\sin x}}{|\cos x| - 1}$

Solutions

1. $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$,
2. $[\frac{1}{2}(3 - \sqrt{13}), \frac{1}{2}(3 - \sqrt{5})] \cup [\frac{1}{2}(3 + \sqrt{5}), \frac{1}{2}(3 + \sqrt{13})]$,
3. $\mathbb{R} \setminus \{\frac{\pi}{2} - 6 + k\pi, k \in \mathbb{Z}\}$,
4. $[0, 1[\cup]1, +\infty[$,
5. $\mathbb{R} \setminus \{\frac{\pi}{3} + k\pi, k \in \mathbb{Z}\}$,
6. $\bigcup_{k\in\mathbb{Z}}]2k\pi, \pi + 2k\pi[$.

Exercise 16. Let us consider the functions

$$f(x) = \cos(\arcsin x), \qquad g(x) = \arcsin(\cos x).$$

Determine their domains and their image sets; establish if they are monotone; if not, find a possible restriction which is invertible.

Solutions

$\text{dom}(f) = [-1,1]$, $f([-1,1]) = [0,1]$; f is not monotone; $f|_{[-1,0]}$ is increasing, $f|_{[0,1]}$ is decreasing, therefore the two restrictions are both invertible; $\text{dom}(g) = \mathbb{R}$, $g(\mathbb{R}) = [-\frac{\pi}{2}, \frac{\pi}{2}]$; g is not monotone; for any $k \in \mathbb{Z}$, $g|_{[2k\pi, \pi+2k\pi]}$ is decreasing, $g|_{[\pi+2k\pi, 2\pi+2k\pi]}$ is increasing, therefore each restriction is invertible.

Exercise 17. Determine if the following functions are even or odd. Then, establish if their graph is symmetric with respect to some straight line or some point of the Cartesian plane.

$$f : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}, \quad f(x) = \sin(2x) + \tan\left(\frac{x}{2}\right);$$
$$i : \mathbb{R} \to \mathbb{R}, \quad i(x) = \arctan\left(\frac{5x}{x^2+2}\right).$$

Solutions

f is odd, since it is the sum of two odd functions, $\sin(2\,\cdot)$ and $\tan(\frac{\cdot}{2})$: its graph is symmetric with respect to the origin of the axes.

i is odd, since it is the composition of two odd functions:

$$i_1(x) = \frac{5x}{x^2+2}, \quad i_2(x) = \arctan x,$$

indeed, for all $x \in \mathbb{R}$, $i(x) = i_2(i_1(x)) = \arctan\left(\frac{5x}{x^2+2}\right)$.

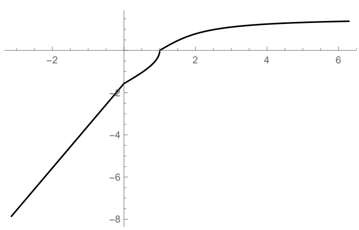
Exercise 18. Consider the following function:

$$f : \mathbb{R} \to]-\infty, -\frac{\pi}{2}[, \quad f(x) = \begin{cases} 2x - \frac{\pi}{2} & \text{if } x < 0, \\ -\arccos x & \text{if } 0 \le x < 1, \\ \arctan(x-1) & \text{if } x \ge 1. \end{cases}$$

(i) Draw the graph of the function f.
(ii) Compute $f(-3)$, $f(0), f(2), f^{-1}(-1), f^{-1}(\frac{\sqrt{3}}{3})$.
(iii) Is f bijective? If it is the case, determine the inverse function f^{-1}.
(iv) Is f^{-1} monotone? Why?

Solutions

(i)



(ii) $f(-3) = -(6 + \frac{\pi}{2})$, $f(0) = -\frac{\pi}{2}$, $f(2) = \frac{\pi}{4}$, $f^{-1}(-1) = \cos(1)$, $f^{-1}(\frac{\sqrt{3}}{3}) = \tan(\frac{\sqrt{3}}{3}) + 1$.
(iii) Yes, the function is bijective, and its inverse f^{-1} is the following piecewise function:

$$f^{-1} :] - \infty, \frac{\pi}{2}[\to \mathbb{R}, \quad f^{-1}(y) = \begin{cases} \frac{y}{2} + \frac{\pi}{4} & \text{if } y < -\frac{\pi}{2}, \\ \cos y & \text{if } -\frac{\pi}{2} \leq y < 0, \\ \tan y + 1 & \text{if } 0 \leq y < \frac{\pi}{2}. \end{cases}$$

(iv) f^{-1} is increasing, since it is the inverse of an increasing function.

Exercise 19. Consider the real-valued function defined as follows:

$$f(x) = \begin{cases} \arctan(2x) & \text{if } x < 0, \\ x^2 + 1 & \text{if } x \geq 0. \end{cases}$$

Determine the inverse function $x = f^{-1}(y)$.

Solution
The images through f of the sets in which f is defined are $f(]-\infty, 0[) =]-\frac{\pi}{2}, 0[$ and $f([0, +\infty[) = [1, +\infty[$. Hence

$$f^{-1} : \left] -\frac{\pi}{2}, 0\right[\cup [1, +\infty[\to \mathbb{R}, \quad f^{-1}(y) = \begin{cases} \dfrac{\tan y}{2} & \text{if } \dfrac{\pi}{2} < y < 0, \\ \\ \sqrt{y-1} & \text{if } y \geq 1. \end{cases}$$