16 Novembre

Citeremo 3 distinte regole dell'Hôpital, due $\frac{0}{0}$ ed una $\frac{\infty}{\infty}$

<u>Teor</u> (1° regola $\frac{0}{0}$). Siano f e g definite in $[a,b]$, sia $x_0 \in [a,b]$, supponiamo che esistono $f'(x_0)$ e $g'(x_0)$ con $g'(x_0) \neq 0$ e sia $f(x_0) = g(x_0) = 0$

Allora
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

<u>Osservazione</u> $\exists\, \delta_0 > 0$ t.c. per $0 < |x - x_0| < \delta_0$ si ha $g(x) \neq 0$.

Infatti $\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0) \neq 0$

posso concludere, dalla conservazione del segno, che $\exists\, \delta_0 > 0$ t.c.

per $0 < |x - x_0| < \delta_0$, $\frac{g(x) - g(x)}{x - x_0}$ ha lo stesso segno di $g'(x_0)$.

$$\frac{g(x)}{x-x_0} \neq 0 \quad \text{per} \quad 0<|x-x_0|<\delta_0 \qquad x\in[a,b]$$

$$\Updownarrow$$

$$g(x) \neq 0 \quad \text{per} \quad 0<|x-x_0|<\delta_0 \qquad x\in[a,b]$$

E pertanto $\dfrac{f(x)}{g(x)}$ è ben definito

per $\quad 0<|x-x_0|<\delta_0 \qquad x\in[a,b]$.

Dim

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{\dfrac{f(x)-f(x_0)}{x-x_0}}{\dfrac{g(x)-g(x_0)}{x-x_0}}$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{\dfrac{f(x)-f(x_0)}{x-x_0}}{\dfrac{g(x)-g(x_0)}{x-x_0}}$$

$$= \frac{f'(x_0)}{g'(x_0)} \qquad \square$$

$(-1, +\infty)$

Esempi

$$\lim_{x \to 0} \frac{\lg(1+x) - 2x}{\sin x}$$

$$\sin(0) = 0$$

$$\lg(1+x) - 2x \Big|_{x=0} = \lg 1 = 0$$

$$(\sin x)'\Big|_{x=0} = \cos x \Big|_{x=0} = 1$$

$$\left(\lg(1+x) - 2x\right)' = = \left(\lg(1+x)\right)' - (2x)'$$

$$= \lg'(1+x)\ (1+x)' - 2$$

$$= \frac{1}{1+x} - 2 \Big|_{x=0} = 1 - 2 = -1$$

Allora

$$\lim_{x \to 0} \frac{\lg(1+x) - 2x}{\sin x} = \frac{-1}{1} = -1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{(\sin x)'(0)}{(x)'(0)} = \frac{1}{1} = 1$$

Teor (2° regola Hôp. $\frac{0}{0}$) Sia I un intervallo e $x_0 \in \overline{\mathbb{R}}$ un punto di accumulazione di I.

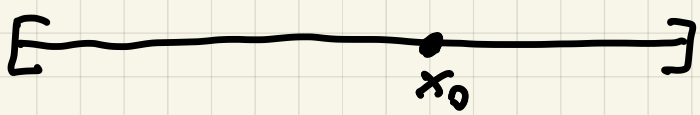
Siano $f, g : I \setminus \{x_0\} \to \mathbb{R}$ derivabili in $I \setminus \{x_0\}$, con $g(x) \neq 0$ e $g'(x) \neq 0$ $\forall\, x \in I \setminus \{x_0\}$. Siano inoltre

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 .$$

Allora, se $\displaystyle\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$

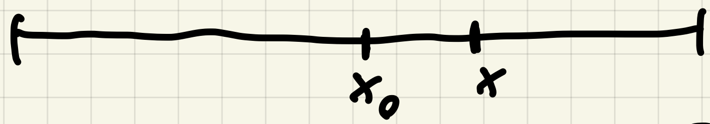
si ha anche $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$

Dim (nel caso $x_0 \in \mathbb{R}$).



Posso estendere in x_0 ponendo $f(x_0) = g(x_0) = 0$ in modo tale che $f(x)$ e $g(x)$ sono continue in x_0.

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)}$$



Per fissare le idee, sia $x > x_0$. Possiamo applicare il teor di Cauchy all'intervallo $[x_0, x]$. Allora esiste $c_x \in (x_0, x)$

t.c. $$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(c_x)}{g'(c_x)} \qquad x_0 < c_x < x$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(c_x)}{g'(c_x)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)} = L$$

Esempio $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2}$

$= \frac{1}{6}\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \dfrac{1}{6}$

$\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{\lg(x^x)} =$

$= \lim\limits_{x\to 0^+} e^{x \lg x} = \lim\limits_{x\to 0^+} e^{\frac{\lg x}{\frac{1}{x}}} = e^0 = 1$

$\lim\limits_{x\to 0^+} \dfrac{\lg x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$

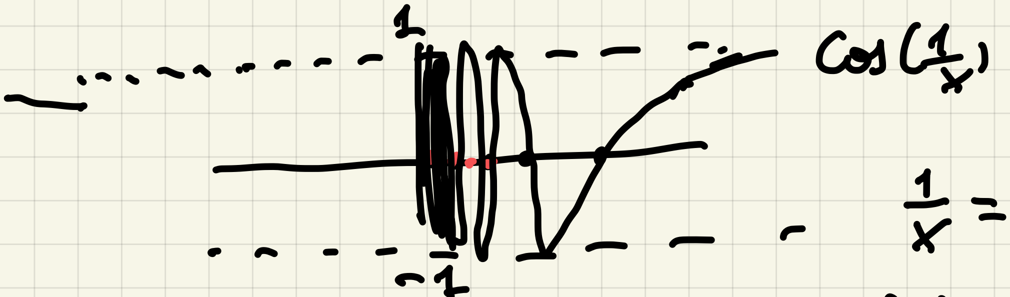
$= -\lim\limits_{x\to 0^+} x = 0$

Es $\quad f(x) = x^2\cos\left(\frac{1}{x}\right), \quad g(x) = x, \quad x>0$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\cancel{x^2}\cos\left(\frac{1}{x}\right)}{\cancel{x}} = \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$$

$$0 \le \underbrace{\left|x\cos\left(\tfrac{1}{x}\right)\right|}_{\to 0} \le |x| \xrightarrow{x\to 0} 0$$

$$\left(\lim_{x\to x_0} f(x) = 0 \iff \lim_{x\to x_0} |f(x)| = 0 \right)$$



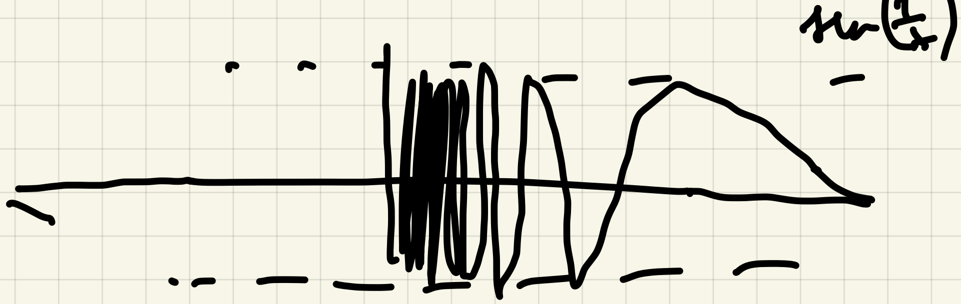
$\frac{1}{x} = \frac{\pi}{2} + k\pi \quad k\in\mathbb{Z}$

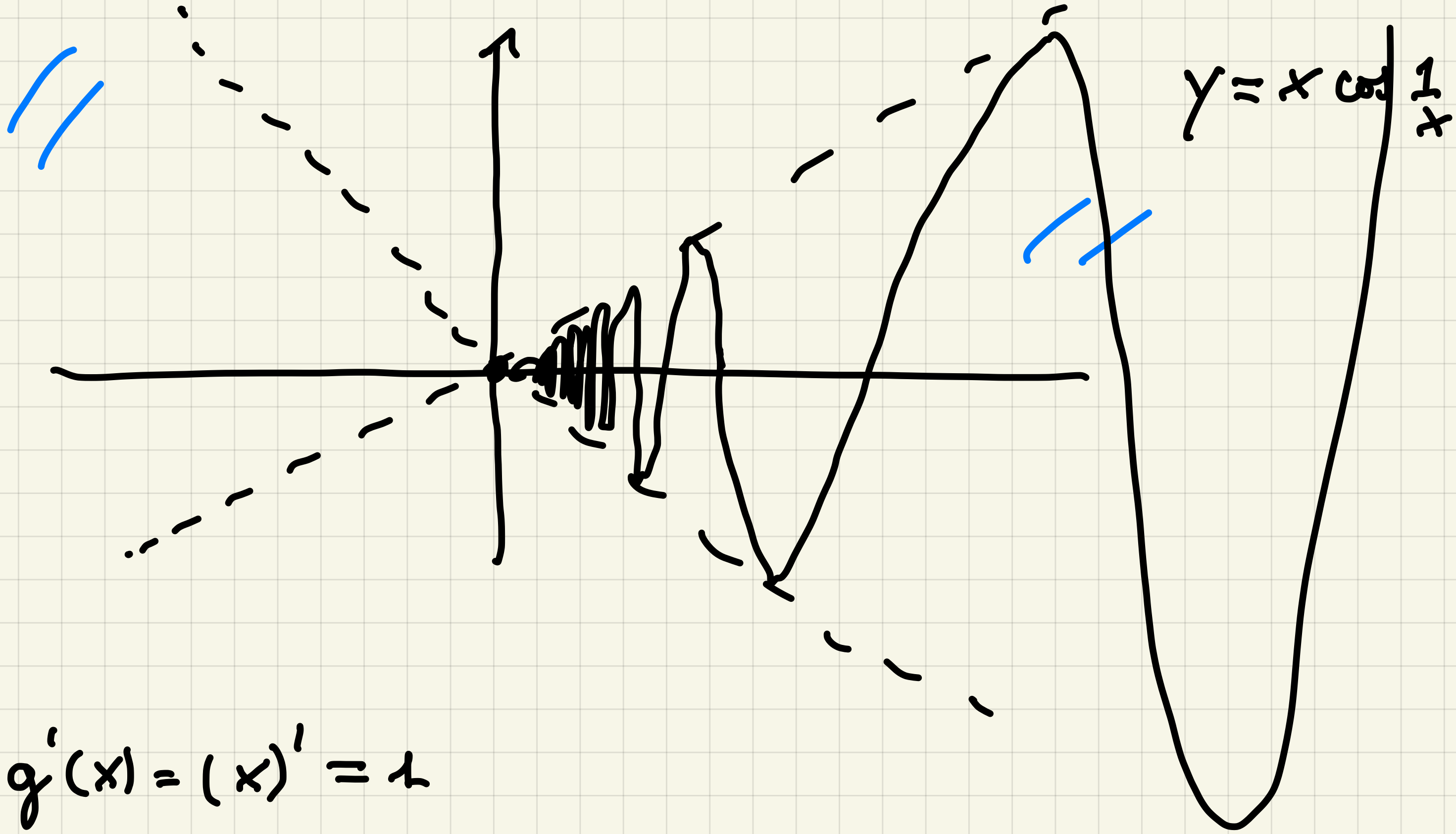
$\cos\left(\frac{1}{x}\right) = \cos\left(\frac{\pi}{2}+k\pi\right) = 0$

$x = \dfrac{1}{\frac{\pi}{2}+k\pi}$

$\frac{1}{x} = 2\pi k \qquad \cos\left(\frac{1}{x}\right) = 1$

$\frac{1}{x} = \pi + 2\pi k \qquad \cos\left(\frac{1}{x}\right) = -1$





$$g'(x) = (x)' = 1$$

$$f'(x) = \left(x^2 \cos\left(\frac{1}{x}\right)\right)' = 2x \cos\left(\frac{1}{x}\right) + x^2 \left(\cos\left(\frac{1}{x}\right)\right)'$$

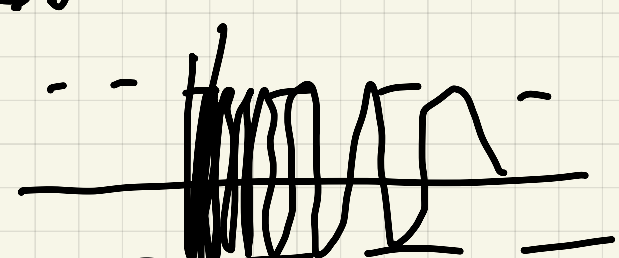
$$= 2x \cos\left(\frac{1}{x}\right) + x^2 \cos'\left(\frac{1}{x}\right) \left(\frac{1}{x}\right)'$$

$$= 2x \cos\left(\frac{1}{x}\right) + \cancel{x^2} \left(\cancel{-} \sin\left(\frac{1}{x}\right)\right) \left(\cancel{-} \frac{1}{\cancel{x^2}}\right)$$

$$= 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \left[\underbrace{2x \cos\left(\frac{1}{x}\right)} + \sin\left(\frac{1}{x}\right)\right]$$

$$= \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{non esiste}$$



Teor (Terza regola $\frac{\infty}{\infty}$) I intervallo, $x_0 \in I'$

Siano $f, g : I \setminus \{x_0\} \to \mathbb{R}$ derivabili in $I \setminus \{x_0\}$.

Si assume che $g'(x) \neq 0 \ \forall x$ in $I \setminus \{x_0\}$.

Siano inoltre

$$\lim_{x \to x_0} f(x) = \pm\infty \quad e \quad \lim_{x \to x_0} g(x) = \pm\infty$$

Allora se esiste $\displaystyle\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$

si ha anche $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = L.$

Esempio
$$\lim_{x \to +\infty} \frac{x^n}{e^x} = \lim_{x \to +\infty} \frac{n\,x^{n-1}}{e^x} =$$

$$= \lim_{x \to +\infty} \frac{n(n-1)\,x^{n-2}}{e^x} = \dots = \lim_{x \to +\infty} \frac{n!}{e^x}$$

$$= 0$$

Conclusione $\quad x^n << e^x \quad$ per $x >> 1$