16 Novembre

$u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d)$, $\exists$ $T > 0$

e corrispondente soluzione

$u \in L^4([0,T], \dot{H}^{\frac{d-1}{2}})$

$u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1})$

$u \in C^0([0, T_{u_0}), \dot{H}^{\frac{d}{2}-1})$

Blow up $\int_0^{T_{u_0}} |u|^4_{\dot{H}^{\frac{d-1}{2}}} dt = +\infty$

se $T_{u_0} < \infty$

Sia per assurdo $T = T_{u_0} < +\infty$ e

$$\int_0^T |u|^4_{\dot{H}^{\frac{d-1}{2}}} dt < \infty$$

$$g(\xi) = \sup_{0 \le t' \le T} |\hat{u}(t', \xi)|$$

Lemma $|\xi|^{\frac{d}{2}-1} g \in L^2(\mathbb{R}^d)$

$$\int_0^T |u|^4_{\dot H^{\frac{d-1}{2}}}\, dt < \infty$$

$$g(\xi) = \sup_{0 \le t' \le T} |\hat{u}(t', \xi)|$$

<u>Lemma</u> $|\xi|^{\frac{d}{2}-1} g \in L^2(\mathbb{R}^d)$

$$\| |\xi|^{\frac{d}{2}-1} g \|_{L^2} = \left(\int_{\mathbb{R}^d} |\xi|^{d-2} \left(\sup_{0 \le t' \le T} |\hat{u}(t', \xi)|^2 \right) d\xi \right)^{\frac{1}{2}}$$

$$\le |u_0|_{\dot H^{\frac{d}{2}-1}} + C_\gamma \, |Q(u,u)|_{L^2([0,T], \dot H^{\frac{d}{2}-2})}$$

$$\le |u_0|_{\dot H^{\frac{d}{2}-1}} + C_\gamma \, |u|^2_{L^4([0,T], \dot H^{\frac{d-1}{2}})} < \infty$$

$\square$

$$g(\xi) = \sup_{0 \le t' \le T} |\hat{u}(t', \xi)|$$

$$|\xi|^{\frac{d}{2}-1} g \in L^2(\mathbb{R}^d)$$

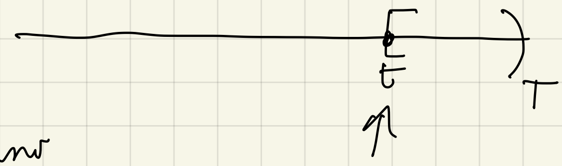
Scelgo ρ t.c. $\int_{|\xi| \ge \rho} |\xi|^{d-2} |g(\xi)|^2 d\xi < \epsilon$

$$\Longrightarrow \boxed{\int_{|\xi| \ge \rho} |\xi|^{d-2} |\hat{u}(t, \xi)|^2 d\xi < \epsilon} \quad \forall \; 0 \le t \le T$$

$$|u(t)|_{\dot H^{\frac{d}{2}-1}} \le |u_0|_{\dot H^{\frac{d}{2}-1}} + C_\gamma \, |u|^2_{L^4([0,T], \dot H^{\frac{d-1}{2}})}$$

$$\forall \; 0 \le t < T$$

Per ogni $t \in [0, T)$ considero il problema con dato iniziale $u(t)$



Scegliendo C opportuno so che esiste un tempo minimo di esistenza τ delle soluzioni, che vivono cioè in $[t, t+\tau]$

$$\left(\frac{\gamma^{\frac{3}{4}}}{8 \rho^{\frac{1}{2}} C \, \| u(t) \|_{\dot{H}^{\frac{d}{2}-1}}} \right)^4 > \tau_1 > 0 \quad \forall t \in [0, T)$$

Quindi, scegliendo $\tau \in (0, \tau_1]$

Restano definite $u(t+s)$ in $[t, t+\tau)$

$\Longrightarrow$ u si estende a $C^0([0, T_{u_0}+\tau), \dot{H}^{\frac{d}{2}-1})$

~~altro~~ $\Longrightarrow$ $u \in L^4([0, S], \dot{H}^{\frac{d-1}{2}})$

per $0 < S < T_{u_0} + \tau$

$$u_0 \in L^3(\mathbb{R}^3) \supset \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$$

__Teor__ Sia $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ u_0 a div nulla. Allora $\exists$ $T > 0$ ed una unica soluzione $\boxed{u \in C^0([0,T], L^3)}$, a div. nulla, di

$$u = e^{\nu t \Delta} u_0 + B(u,u)$$

Inoltre $\exists$ $\varepsilon_{3\nu} > 0$ t.c. se $|u_0|_{L^3} < \varepsilon_{3\nu}$ si ha $T = +\infty$

Se inoltre $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3)$ i tempi di esistenza nei due teoremi sono gli stessi.

$$X = C^0([0,T], L^3)$$

$$\cancel{B : X \times X \to X}$$

Def (Spazi di Kato). Per $p \in [d, +\infty]$ e $T \in (0, \infty)$

$$K_p(T) = \left\{ u \in C^0((0,T], L^p(\mathbb{R}^d, \mathbb{R}^d)) \; : \; |u|_{K_p(T)} = \sup_{t \in (0,T]} t^{\frac{d}{2}\left(\frac{1}{d} - \frac{1}{p}\right)} |u(t)|_{L^p} < \infty \right\}$$

$$K_d \supseteq C^0([0,T], L^d)$$

Per $p \in [1, d)$

$$K_p(T) = \left\{ u \in C^0([0,T], L^p) \; : \; |u|_{K_p(T)} = \sup_{t \in [0,T]} t^{\frac{d}{2}\left(\frac{1}{d} - \frac{1}{p}\right)} |u(t)|_{L^p} < \infty \right\}$$

$T = \infty$ $\qquad (0,T] \rightsquigarrow (0, \infty)$

$\qquad\qquad [0,T] \rightsquigarrow [0, \infty)$

$u_0 \in L^d(\mathbb{R}^d)$ $\qquad p \geq d$

$$\underbrace{\left| e^{t\Delta} u_0 \right|_{L^p(\mathbb{R}^d)}} \leq (4\pi t)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{d}\right)} |u_0|_{L^d}$$

$$e^{t\Delta} u_0 \in C^0\left(\mathbb{R}_+, L^p(\mathbb{R}^d)\right)$$

$$\underbrace{\sup_{\mathbb{R}_+} t^{\frac{d}{2}\left(\frac{1}{d}-\frac{1}{p}\right)} \left|e^{t\Delta} u_0\right|_{L^p}}_{\cancel{K} \quad \left| e^{t\Delta} u_0 \right|_{K_p(\infty)}} \leq C \, |u_0|_{L^d}$$

$K_p(T) \quad \forall T.$

$e^{t\Delta} u_0 \in K_p(\infty)$

<u>Lemma</u> $u_0 \in L^d(\mathbb{R}^d, \mathbb{R}^d)$ e se $p > d$ allora

$$\lim_{T \to 0} \left| \boxed{e^{t\Delta} u_0} \right|_{K_p(T)} = 0 \qquad *$$

<u>Dim</u> ~~Sia~~ $\forall \varepsilon > 0 \ \exists \ \phi \in L^d \cap L^p$ t.c.

$|\phi - u_0|_{L^d} \leq \varepsilon$. $\qquad e^{t\Delta} u_0 = e^{t\Delta}\phi + e^{t\Delta}(u_0 - \phi)$

$$\left| e^{t\Delta}(u_0 - \phi) \right|_{K_p(T)} \leq C_p \, |u_0 - \phi|_{L^d} < C_p \, \varepsilon$$

$$\left| e^{t\Delta} \phi \right|_{K_p(T)} = \sup_{[0,T]} t^{\frac{d}{2}\left(\frac{1}{d}-\frac{1}{p}\right)} \left| e^{t\Delta}\phi \right|_{L^p} \leq \sup_{(0,T)} t^{\frac{d}{2}\left(\frac{1}{d}-\frac{1}{p}\right)} |\phi|_{L^p}$$

$$= T^{\frac{d}{2}\left(\frac{1}{d}-\frac{1}{p}\right)} |\phi|_{L^p}$$

Lemma Siano p, q, r con

$$0 < \frac{1}{p} + \frac{1}{q} \leq 1$$

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} < \frac{1}{d} + \frac{1}{r}$$

Allora $B : K_p(T) \times K_q(T) \to K_r(T)$ ed esiste C_{pqr} indipendente da T, t.c.

$$\|B(u,v)\|_{K_r(T)} \leq C \|u\|_{K_p(T)} \|v\|_{K_q(T)} .$$

$\forall u_0 \in L^d(\mathbb{R}^d, \mathbb{R}^d)$ fissato

Corollario $\forall\ p \in (d, \infty]$ $\exists$ $\varepsilon_p > 0$ t.c.

se $\| e^{t\Delta} u_0 \|_{K_p(T)} < \varepsilon_p$ allora esiste ed è unica una soluzione $u \in K_p(T)$ con $\|u\|_{K_p(T)} < 2\varepsilon_p$, di

$$u = e^{t\Delta} u_0 + B(u,u).$$

Dim Se $p = q = r$, $p > d$ $\quad B : K_p(T) \times K_p(T) \to K_p(T)$ $\|$

$$\varepsilon_p < \frac{1}{4\|B\|}$$

Teor $\forall\, u_0 \in L^d(\mathbb{R}^3,\mathbb{R}^3)$ $\exists\, T>0$ ed una soluzione $u \in C^0([0,T), L^3)$ ed è unico. Inoltre se $|u_0|_{L^3} < \varepsilon_3$ per $\varepsilon_3 > 0$ opportuno, allora $T = +\infty$.

Dim Supponiamo che esiste una soluzione $u \in K_{2d}(T)$

$$u = \underline{e^{t\Delta} u_0} + B(u,u)$$

$$B : K_{2d}(T) \times K_{2d}(T) \longrightarrow C^0([0,T], L^d) \cap K_d(T)$$

$p = q = 2d$

$r = d$

$$0 < \underbrace{\frac{1}{p} + \frac{1}{q}}_{\frac{1}{d}} < 1 \qquad\qquad \frac{1}{r} \le \frac{1}{p} + \frac{1}{q} < \frac{1}{d} + \frac{1}{r}$$

$$\frac{1}{d} \le \frac{1}{d} < \frac{1}{d} + \frac{1}{r}$$

$$u \in C^0([0,T], L^3)$$

p

se $|u_0|_{L^3} < \varepsilon \Rightarrow |e^{t\Delta} u|_{K_{2d}(\infty)} \le C\varepsilon$ $\quad u \in K_{2d}(\infty)$

$u_{21} = u_2 - u_1$

$$\boxed{w_j = B(u_j, u_j)}$$

$$\partial_t u_j - \Delta u_j = Q(u_j, u_j)$$

$$\partial_t u_{21} - \Delta u_{21} = Q(u_2, u_2) - Q(u_1, u_1) =$$
$$= Q(u_2 + u_1, u_{21})$$

$$\boxed{u_1 = e^{t\Delta} u_0 + w_1 \quad , \quad u_2 = e^{t\Delta} u_0 + w_2}$$

$$\partial_t u_{21} - \Delta u_{21} = f_{21} = Q(u_2 + u_1, u_{21}) =$$

$$\boxed{\partial_t u_{21} - \Delta u_{21} = 2Q(e^{t\Delta} u_0, u_{21}) + Q(w_1, u_{21}) + Q(w_2, u_{21})} \quad *$$

$$u_{21}|_{t=0} = 0$$

Utilizzo la stima $\qquad \dot{H}^{-\frac{1}{2}} \supset L^{\frac{3}{2}} \; , \; L^3 \supset \dot{H}^{\frac{1}{2}}$

$$|Q(u,v)|_{\dot{H}^{-\frac{3}{2}}} = |\operatorname{div}(u \otimes v)|_{\dot{H}^{-\frac{3}{2}}} \lesssim |u \otimes v|_{\dot{H}^{-\frac{1}{2}}}$$

$$\lesssim |u \otimes v|_{L^{\frac{3}{2}}} \leq |u|_{L^3} \; |v|_{L^3}$$

$$C^0([0,T], L^3)$$

$$Q(u,v) \in C^0([0,T], \dot{H}^{-\frac{3}{2}}) \subseteq L^2([0,T], \dot{H}^{-\frac{3}{2}})$$

$$\boxed{\partial_t u_{21} - \Delta u_{21} = 2\, Q(e^{t\Delta}u_0, u_{21}) + Q(w_1, u_{21}) + Q(w_2, u_{21})}$$

$$L^2([0,T], \dot{H}^{-\frac{3}{2}})$$

$$|u_{21}|^2_{\dot{H}^{-\frac{1}{2}}} + 2\nu \int_0^t |\nabla u_{21}|^2_{\dot{H}^{-\frac{1}{2}}}\, dt' = 2 \int_0^t \langle f_{21}, u_{21} \rangle_{\dot{H}^{-\frac{1}{2}}}\, dt'$$

$$\leq 4 \int_0^t |Q(e^{t\Delta}u_0, u_{21})|_{\dot{H}^{-\frac{3}{2}}}\, |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}}\, dt'$$

$$+ 2 \int_0^t |Q(w_1, u_{21}) + Q(w_2, u_{21})|_{\dot{H}^{-\frac{3}{2}}}\, |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}}\, dt'$$

$$2 \int_0^t |Q(w_j, u_{21})|_{\dot{H}^{-\frac{3}{2}}}\, |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}}\, dt' \leq$$

$$\lesssim \int_0^t |w_j|_{L^3}\, |u_{21}|_{L^3}\, |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}}\, dt' \leq$$

$$\lesssim |w_j|_{K_3(t)} \int_0^t |\nabla u_{21}|^2_{\dot{H}^{-\frac{1}{2}}}\, dt$$

$$|u_{21}|^2_{\dot{H}^{-\frac{1}{2}}} \leq C \int_0^t |u_{21}|^2_{\dot{H}^{-\frac{1}{2}}}\, dt'$$

$$\Longrightarrow |u_{21}|_{\dot{H}^{-\frac{1}{2}}} \equiv 0 \text{ in } [0,t]$$

cioè $\mathcal{I} = \{ t : u_{21}(t) = 0,\ t \in [0,T] \}$ è aperto

Il fatto $\mathcal{I}$ è chiuso, poiché $u_{21} \in C^0([0,T], L^3)$

e che $0 \in \mathcal{I} \Longrightarrow \mathcal{I} = [0,T]$.