RANGO

$A \in M_{m,n}(\mathbb{K})$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(m)} \end{pmatrix} \qquad A^{(i)} \in \mathbb{K}^n$$

Se A è a gradini allora $\operatorname{rg} A = \#\{\text{pivot}\} = \#\{\text{righe} \neq 0\}$

$$\begin{array}{c} \begin{matrix} & & \#^{\circ} & & & & \end{matrix} \\ \begin{pmatrix} 0 \cdots 0 & a_1 & \cdots & \cdots & \cdots & \cdots \\ 0 \cdots 0 & 0 & \cdots\, 0 & a_2 & \cdots & \cdots \\ 0 \cdots & & 0 & 0 \cdots 0 & a_3 & \cdots \\ \cdots & & & & & \\ 0 \cdots & & 0 & \cdots\, 0 & a_k & \cdots \\ 0 \cdots & & & & & \end{pmatrix} \end{array} \qquad a_i \neq 0 \quad \forall i$$

OSS. chiave: le operazioni elementari sulle righe non modificano il rango

1) Scambio di righe ✓

2) $A^{(i)} \rightsquigarrow \lambda A^{(i)}$ per un certo i, $\lambda \neq 0$ ✓

3) $A^{(i)} \rightsquigarrow A^{(i)} + \lambda A^{(j)} = (A^{(i)})'$ ✓

$$A^{(i)} = (A^{(i)})' - \lambda A^{(j)}$$

Es.

$$1 \quad 2 \quad 3$$

$$\begin{vmatrix} 2 & 1 & 3 \end{vmatrix}$$

$$\longrightarrow A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\boxed{\operatorname{rg} A = 3} \Longrightarrow A \text{ invertibile}$$

Operazioni elementari sulle righe (riviste)

$$A \in M_{m,n}(\mathbb{K}) \qquad I_m = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in M_m(\mathbb{K})$$

E = op. element. su A $\rightsquigarrow$ E applicata a I_m

$E(A) = A'$ $\qquad$ $E(I_m)$

$$\boxed{E(A) = E(I_m) \cdot A}$$

$$B \cdot A = \begin{pmatrix} BA_{(1)} & BA_{(2)} & \cdots & BA_{(m)} \end{pmatrix}$$

E = scambio delle i-esima e j-esima riga

$$A = \begin{pmatrix} A^{(1)} \\ \vdots \\ A^{(i)} \\ \vdots \\ A^{(j)} \\ \vdots \\ A^{(m)} \end{pmatrix} \rightsquigarrow E(A) = \begin{pmatrix} A^{(1)} \\ \vdots \\ A^{(j)} \\ \vdots \\ A^{(i)} \\ \vdots \\ A^{(m)} \end{pmatrix}$$

$$E(I_m) = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots \\ \cdots & & & & \cdots \\ 0 & \cdots & 0 & 1 & 0 \cdots 0 \\ \cdots & & & & \cdots \\ 0 & \cdots & 1 & 0 & \cdots 0 \\ \cdots & & & & \cdots \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

i-esimo

j-esimo

$$\boxed{E(I_m) \cdot A = E(A)}$$

$$E(I_m) \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_i \\ \vdots \\ u_m \end{pmatrix}$$

$$E(I_m) \cdot A = \begin{pmatrix} E(I_m) \cdot A_{(1)} & \cdots & E(I_m) A_n \end{pmatrix} = E(A)$$

Inversa di una matrice $A \in GL_n(\mathbb{K})$ = matrici $n \times n$ invertibili

$rg\, A = n$

$$\left(A \mid I_n \right) \in M_{n,2n}(\mathbb{K})$$

$\rightsquigarrow$ Gauss – Jordan

$rg\, A' = rg\, A = n$

$\left(A' \mid B \right)$ a gradini

A' ha n pivot

$$A' = \begin{pmatrix} a_1 & - & - & - \\ 0 & a_2 & - & - \\ - & - & & - \\ 0 & \cdots & 0 & a_n \end{pmatrix} \qquad a_i \neq 0$$

$a_n \rightsquigarrow 1$

Lavorando a ritroso (dall'ultima riga)

$\rightsquigarrow A'' = I_n \qquad \left(I_n \mid C \right)$

$$\left(A' \mid B\right) \xrightarrow[\text{qualunque}]{\text{normal} + \text{tondo}} \left(\begin{matrix} 1 & \cdots & & \\ 0 & 1 & \cdots & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 \end{matrix} \;\middle|\; B'' \right)$$

$$\xrightarrow{\text{op. su righe}} \left(\begin{matrix} 1 & \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \cdots & & \\ 0 & 0 \cdots & 1 \end{matrix} \;\middle|\; B''' \right)$$

$$\longrightarrow \left(I_n \mid C \right) \qquad C = A^{-1}$$

$$\left(A \mid I_n\right) \xrightarrow{k \text{ op. element.}} \cdots \longrightarrow \left(I_n \mid A^{-1}\right)$$

E_i element.

$$\underbrace{E_k E_{k-1} \cdots E_1}_{C} \left(A \mid I_n\right) = C \left(A \mid I_n\right) = \left(CA \mid C\right)$$

Matrici elementari.

$E(I_n)$

$rg\, I_n = n$

matr. el. $\in GL_n(k)$

$CA = I_n \Rightarrow C = A^{-1}$

$$\left(I_n \mid C\right) = \left(CA \mid C\right)$$

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & \underline{-2} & 5 & 0 & -2 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & -2 & 1 \end{array}\right) \rightsquigarrow$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{5}{12} & \frac{1}{6} & -\frac{1}{12} \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{5}{12} & \frac{1}{6} & -\frac{1}{12} \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{array}\right) ; \quad A^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{5}{12} & \frac{1}{6} & -\frac{1}{12} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$\boxed{A A^{-1} = I_3}$$

$$\begin{array}{ccc} \mathcal{B} & & \mathcal{C} \\ f : V & \longrightarrow & W \end{array}$$ lineare $\quad \ker f = ?$

$\ker f = \{ v \in V \mid f(v) = 0_W \}$

Supponiamo di avere $\mathcal{B} = (v_1, \dots, v_n)$, $\mathcal{C} = (w_1, \dots, w_m\}$ basi

$A = M^{\mathcal{C}}_{\mathcal{B}}(f) \in M_{m,n}(\mathbb{K}) \rightsquigarrow L_A : \mathbb{K}^n \longrightarrow \mathbb{K}^m$

$\boxed{\ker L_A \cong \ker f}$

$\ker L_A = \{ X \in \mathbb{K}^n \mid AX = 0 \}$

$\boxed{AX = 0} \rightsquigarrow$ base

$u_1, \dots, u_k \in V$ sono lin. dip. o indip. ?

Supponiamo di avere una base $\mathcal{B} = (v_1, \dots, v_n)$ di V

$$u_i = a_{i1} v_1 + \dots + a_{in} v_n \qquad i = 1, \dots, k$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \qquad A^{(i)} = \text{comp. di } u_i \qquad A \in M_{k,n}(\mathbb{K})$$

$u_1, \dots, u_k$ lin. dip. $\iff$ rg $A < k$

" " indip $\iff$ rg $A = k$

Se $u_1, \dots, u_k$ sono lin. dip. $\exists\ \lambda_1, \dots, \lambda_k \in \mathbb{K}$ non tutti nulli

t.c. $\lambda_1 u_1 + \dots + \lambda_k u_k = 0$ $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix}$ è soluz. del sistema

$${}^tA \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = 0 \qquad {}^tA \in M_{n,k}(\mathbb{K})$$

Si trovano tutte le relazioni di ~~dip.~~ dip. lineare tra

$u_1, \dots, u_k$

$A \in M_{m,n}(\mathbb{K})$ $\quad\quad$ $\operatorname{rg} A = k \leq \min(m,n)$

$\operatorname{rg} A \rightsquigarrow$

$A \rightsquigarrow$