18 Novembre

$$u_{21} = u_2 - u_1 \quad , \qquad u_j = e^{t\Delta} u_0 + w_j \qquad w_j = B(u_j, u_j)$$

$$\begin{cases} \partial_t u_{21} - \Delta u_{21} = f_{21} \\ f_{21} = 2Q(e^{t\Delta} u_0, u_{21}) + Q(w_2, u_{21}) + Q(w_1, u_{21}) \end{cases}$$

$$|u_{21}(t)|^2_{\dot{H}^{-\frac{1}{2}}} + 2\int_0^t |\nabla u_{21}|^2_{\dot{H}^{-\frac{1}{2}}} dt' = 2 \int_0^t \langle f_{21}, u_{21} \rangle_{\dot{H}^{-\frac{1}{2}}}$$

$$\leq 4 \int_0^t |Q(e^{t'\Delta} u_0, u_{21})|_{\dot{H}^{-\frac{3}{2}}} \; |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}} \, dt'$$

$$+ 2 \int_0^t |Q(w_2, u_{21}) + Q(w_1, u_{21})|_{\dot{H}^{-\frac{3}{2}}} \; |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}} \, dt'$$

$$\lesssim (|w_1|_{K_3(t)} + |w_2|_{K_3(t)}) \int_0^t |\nabla u_{21}|^2_{\dot{H}^{-\frac{1}{2}}} dt'$$

$$|Q(u, v)|_{\dot{H}^{-\frac{3}{2}}} \lesssim |u|_{L^3} \, |v|_{L^3}$$

$$\int_0^t |Q(e^{t'\Delta} u_0, u_{21})|_{\dot{H}^{-\frac{3}{2}}} \; |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}} \, dt'$$

$$u_0 = u_0^{(1)} + u_0^{(2)} \qquad u_0^{(2)} \in L^3 \cap L^6$$

$$|u_0^{(1)}|_{L^3} < \varepsilon$$

$$\int_0^t |Q(e^{t'\Delta} u_0^{(1)}, u_{21})|_{\dot{H}^{-\frac{3}{2}}} \; |\nabla u_{21}|_{\dot{H}^{-\frac{1}{2}}}$$

$$\lesssim \int_0^t |e^{t'\Delta} u_0^{(1)}|_{L^3} \; \cancel{|u_{21}|_{L^3}} \; |\nabla u_{21}|^2_{\dot{H}^{-\frac{1}{2}}}$$

$$\leq |u_0^{(1)}|_{L^3} \int_0^t |\nabla u_{21}|^2_{\dot{H}^{-\frac{1}{2}}}$$

$$\int_0^t |Q(e^{t'\Delta}u_0^{(2)}, u_{21})|_{\dot H^{-\frac32}} \, |\nabla u_{21}|_{\dot H^{-\frac12}} \, dt'$$

$$= \int_0^t |\cancel{\nabla}(e^{t'\Delta}u_0^{(2)} u_{21})|_{\dot H^{\cancel{-\frac32}} -\frac12} \, |\nabla u_{21}|_{\dot H^{-\frac12}} \, dt' \qquad \dot H^{-\frac12} > L^{\frac32}$$

$$\le \int_0^t |e^{t'\Delta}u_0^{(2)} u_{21}|_{L^{\frac32}} \, |\nabla u_{21}|_{\dot H^{-\frac12}} \, dt' \qquad \frac23 = \frac16 + \frac12$$

$$\le \int_0^t |e^{t'\Delta}u_0^{(2)}|_{L^6} \, |u_{21}|_{L^2} \, |\nabla u_{21}|_{\dot H^{-\frac12}} \, dt'$$

$$\le |u_0^{(2)}|_{L^6} \int_0^t |u_{21}|^{\frac12}_{\dot H^{-\frac12}} \, \cancel{|u_{21}|^{\frac12}_{\dot H^{\frac12}}} \, |\nabla u_{21}|^{\frac32}_{\dot H^{-\frac12}} \, dt'$$

$$\frac34 + \frac14 = 1$$

$$\le |u_0^{(2)}|_{L^6} \left(C_\varepsilon \int_0^t |u_{21}|^2_{\dot H^{-\frac12}} \, dt' + \varepsilon \int_0^t |\nabla u_{21}|^2 \, dt' \right)$$

$$|u_{21}(t)|^2_{\dot H^{-\frac12}} + \cancel{c\int_0^t |\nabla u_{21}|^2_{\dot H^{-\frac12}}} \le C_\varepsilon |u_0^{(2)}|_{L^6} \int_0^t |u_{21}|^2_{\dot H^{-\frac12}} \, dt'$$

$\exists$ $t_0 > 0$ t.c. $\forall$ $t \in [0, t_0]$ si ha

$\exists C_1'$

$$|u_{21}(t)| \le C_1' \int_0^t |u_{21}|_{\dot H^{-\frac12}} \, dt'$$

$$\Longrightarrow \quad u_{21} \equiv 0 \quad \text{in } [0, t_0]$$

$$\Longrightarrow \quad X = \{ t \in [0,T] : u_{21} \equiv 0 \text{ in } [0,t] \}$$

X è aperto in $[0,T]$.

ed è un chiuso, $u_{21} \in C^0([0,T], L^3)$

$X = [0,T]$, $u_{21} \equiv 0$ in $[0,T]$.

$$(\partial_t - \Delta) B(u,v) = Q(u,v) \qquad B(u,v)_{|t=0} = 0$$

Lemma $\quad 0 < \frac{1}{p} + \frac{1}{q} \leq 1$

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} < \frac{1}{d} + \frac{1}{r}$$

$$|B(u,v)|_{K_r(T)} \leq C\, |u|_{K_p(T)}\, |v|_{K_q(T)}$$

$$\begin{cases} (\partial_t - \Delta) L_m f = \mathbb{P}\, \partial_m f \\ L_m f_{|t=0} = 0 \end{cases}$$

$$\widehat{\mathbb{P} u}^i = c_{ijkl} \frac{\xi_j \xi_k}{|\xi|^2} \hat{u}^l$$

$$\hat{u}^i - \frac{\xi_i \xi_j}{|\xi|^2} \hat{u}^j$$

$$\widehat{L_m f}^i = c_{ijkl} \int_0^t e^{-(t-t')|\xi|^2} \frac{\xi_j \xi_k \xi_m}{|\xi|^2} \hat{f}^l(\xi)\, dt'$$

$$(L_m f)^i = c_{ijkl} \int_0^t \Gamma_{jkm}(t-t') * f^l \, dt'$$

$$Q(u,v) = -\mathbb{P}\, \mathrm{div}(u \otimes v) = -\mathbb{P}\, \partial_m (u \otimes v)_m$$

$$\underline{B(u,v)} = -\boxed{L_m[(u \otimes v)_m]}$$

Lemma $\Gamma_{jkm}(t) \in C^\infty(\mathbb{R}^d)$ ed $\exists\, C > 0$

$$\left|\Gamma_{jkm}(t,x)\right| \le C\left(\sqrt{t}+|x|\right)^{-d-1}$$

$$\hat{\Gamma}_{jkm}(t,\cdot)(\xi) = \frac{\xi_j\xi_k\xi_m}{|\xi|^2}\, e^{-t|\xi|^2}$$

Dim $\Gamma_{jkm}(t,\cdot) \in C^\infty(\mathbb{R}^d) \cap BC(\mathbb{R}^d)$

$$\Gamma_{jkm}(t,x) = \int e^{-i\frac{x}{\sqrt{t}}\cdot\xi\sqrt{t}}\, \frac{\sqrt{t}\xi_j\sqrt{t}\xi_k\sqrt{t}\xi_m}{t|\xi|^2}\, e^{-t|\xi|^2}\, \frac{d(\xi\sqrt{t})}{t^{\frac{1}{2}}\, t^{\frac{d}{2}}}$$

$$= t^{-\frac{d+1}{2}}\, \Gamma_{jkm}\left(1, \frac{x}{\sqrt{t}}\right)$$

$$\boxed{\left|\Gamma_{\cancel{jkm}}(1,x)\right| \le C\left(1+|x|\right)^{-d-1} \quad \forall x}$$

$$\Rightarrow \quad \left|\Gamma(t,x)\right| = t^{-\frac{d+1}{2}}\left|\Gamma\left(1,\frac{x}{\sqrt{t}}\right)\right| \le$$

$$\le C\, t^{-\frac{d+1}{2}}\left(1+\frac{|x|}{\sqrt{t}}\right)^{-d-1}$$

$$= C\left(\sqrt{t}+|x|\right)^{-d-1}$$

$$|\Gamma(x)| \leq C \,(1+|x|)^{-d-1} \qquad (\cancel{\forall})$$

$\Gamma \in BC(\mathbb{R}^3)$. Basta dimostrare $*$ per $|x| >> 1$.

$$\Gamma(x) = \int_{\mathbb{R}^d} e^{-i\xi x} \frac{\xi_j \xi_k \xi_m}{|\xi|^2} e^{-|\xi|^2} d\xi$$

$\chi_0 \in C_c^\infty(\mathbb{R}^d, [0,1])$ $\quad \chi_0 \equiv 1$ vicino a 0.

$$1 = \chi_0 + (1-\chi_0)$$

$$\Gamma(x) = \int_{\mathbb{R}^d} e^{-i\xi x} \frac{\xi_j \xi_k \xi_m}{|\xi|^2} \chi_0(|x|\xi)\, e^{-|\xi|^2} d\xi \qquad \mathrm{I}$$

$$+ \int_{\mathbb{R}^d} e^{-i\xi x} \frac{\xi_j \xi_k \xi_m}{|\xi|^2} (1-\chi_0(|x|\xi))\, e^{-|\xi|^2} d\xi \qquad \mathrm{II}$$

Qui $|x| >> 1$

$$|\mathrm{I}| \leq \int_{|\xi| \leq \frac{c}{|x|}} |\xi| \, d\xi = C_d \, |x|^{-d-1}$$

$$II = \int_{\mathbb{R}^d} e^{-ix\xi} \frac{\xi_i \xi_j \xi_k}{|\xi|^2} e^{-|\xi|^2} \left(1-\chi_0(|x|\xi)\right) d\xi$$

$$II_1 = \int_{\mathbb{R}^d} e^{-ix\xi} \frac{\xi_i \xi_j \xi_k}{|\xi|^2} e^{-|\xi|^2} \overbrace{\chi_0(\xi)}^{B(0,1)} \overbrace{\left(1-\chi_0(|x|\xi)\right)}^{B^c(0,\frac{1}{|x|})} d\xi$$

$$II_2 = \int_{\mathbb{R}^d} e^{-ix\xi} \frac{\xi_i \xi_j \xi_k}{|\xi|^2} e^{-|\xi|^2} \underbrace{\left(1-\chi_0(\xi)\right)}_{B^c(0,1)} \underbrace{\left(1-\chi_0(|x|\xi)\right)}_{B^c(0,\frac{1}{|x|})} d\xi$$

II_2

$$\chi_0 = \begin{cases} 1 & \text{in } B(0,1) \\ 0 & \text{in } B^c(0,2) \end{cases}$$

$$\mathrm{II} = \int e^{-ix\xi} \left(1 - \chi_0(|x|\xi)\right) \frac{\xi^3}{|\xi|^2} e^{-|\xi|^2} d\xi$$

$$L e^{-ix\xi} = e^{-ix\xi}$$

$$L = \frac{1}{-ix} \frac{d}{d\xi}$$

$$i \frac{x}{|x|^2} \cdot \nabla_\xi e^{-ix\xi} =$$

$$= i \frac{x}{|x|^2} \cdot -ix \, e^{-ix\xi}$$

$$= e^{-ix\xi}$$

$$L = i \frac{x}{|x|^2} \cdot \nabla_\xi$$

$$\mathrm{II} = \int \cancel{L^N}\left(e^{-ix\xi}\right) (L^*)^N \left(\left(1 - \chi_0(|x|\xi)\right) \frac{\xi^3}{|\xi|^2} e^{-|\xi|^2} \right)$$

$$L^* = -L$$

$$\left| L^N \left(\left(1 - \chi_0(|x|\xi)\right) \frac{\xi^3}{|\xi|^2} \cancel{e^{-|\xi|^2}} \right) \right|$$

$|\xi| \le 1$

$$\lesssim C_N \frac{1}{|x|^N} |\xi|^{-N+1} \quad |\xi| \gtrsim 1$$

$$\chi_0 = \begin{cases} 1 & \text{in } B(0,1) \\ 0 & B^c(0,2) \end{cases}$$

$$\left| \partial_\xi^\alpha \left(1 - \chi_0(|x|\xi)\right) \right| \lesssim |\xi|^{-|\alpha|}$$

$$- \left| |x|^{|\alpha|} \left(\partial_\xi^\alpha \chi_0\right)(|x|\xi) \right| \lesssim |\xi|^{-|\alpha|}$$

$$1 \le |x||\xi| \le 2 \qquad |x|^{|\alpha|} \sim |\xi|^{-|\alpha|}$$

$$|II| \lesssim \frac{1}{|x|^N} \int_{|\xi| \geq \frac{1}{|x|}} |\xi|^{-N+1} \, d\xi$$

$$\approx \frac{1}{|x|^N} \int_{\frac{1}{|x|}}^{\infty} r^{d-\cancel{1}+\cancel{1}-N} \, dr$$

$$= \frac{1}{|x|^N} \; r^{d+1-N} \Big|_{r=\frac{1}{|x|}}$$

$$= \cancel{\frac{1}{|x|^N}} \left(\frac{1}{|x|}\right)^{d+1} \cancel{\left(\frac{1}{|x|}\right)^{-N}}$$

$$= |x|^{-d-1}$$

I, II li ho maggiorati

$$C \, |x|^{-d-1}$$

$$\boxed{|\Gamma(\cancel{x})| \leq C \, (\cancel{x+}|\cancel{x}|)^{-d-1}}$$

$B(u,v)$