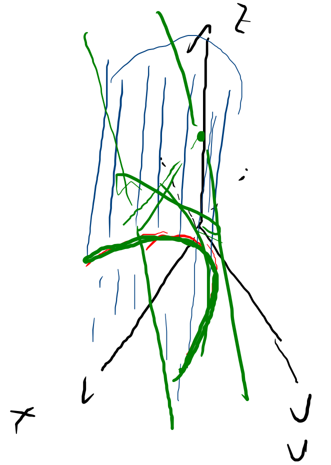
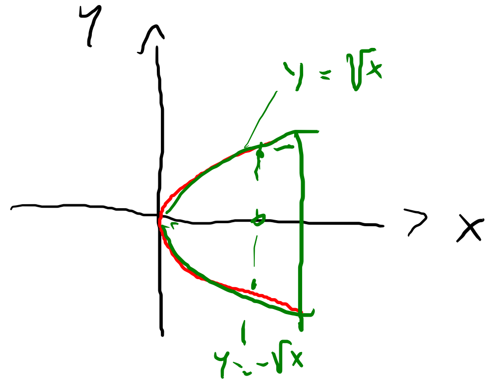
$x = y^2$

$z = 0$

$x + z = 1$

$z = 1 - x$



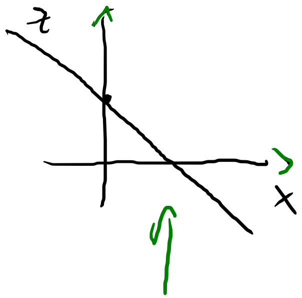


$x = y^2$

$\{(x,y)^T : y \in [-1,1],\ y^2 \le x \le 1\}$

E

Devo calcolare l'integrale su $\{(x,y)^T \in \mathbb{R}^2 : x \in [0,1],\ -\sqrt{x} \le y \le \sqrt{x}\}$ della funzione $f(x,y) = 1 - x$



$x + z = 1$ $\quad z = 1 - x$

$z \ge 0 \Rightarrow x \le 1$

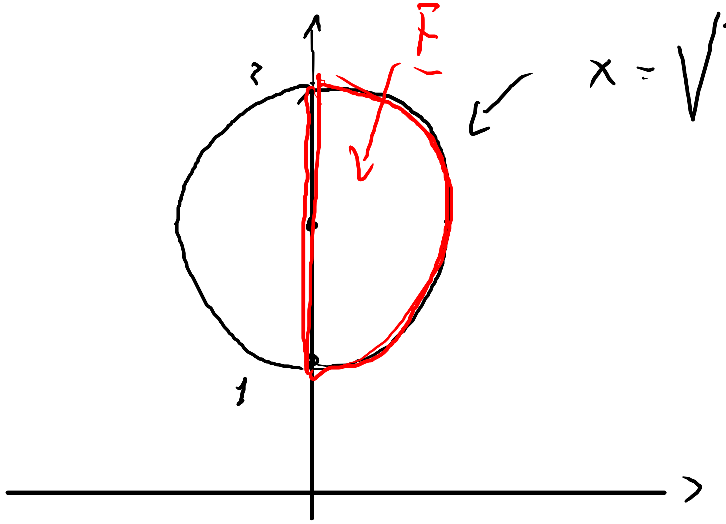
$$\text{Volume} = \iint_E (1-x)\,dx\,dy = \int_{-1}^{1} \left(\int_{y^2}^{1} (1-x)\,dx \right) dy = \int_{-1}^{1} \left[x - \frac{1}{2}x^2 \right]_{y^2}^{1} dy = \int_{-1}^{1} \left(1 - \frac{1}{2} - y^2 + \frac{1}{2}y^4\right) dy$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - y^2 + \frac{1}{2}y^4\right) dy = \left[\frac{1}{2}y - \frac{1}{3}y^3 + \frac{1}{10}y^5\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = 1 - \frac{2}{3} + \frac{1}{5} = \frac{5+3}{15} = \frac{8}{15}$$

$$\iint_E \frac{x}{y}\,dx\,dy$$

$x \geq 0$

$$\boxed{x^2+(y-2)^2 \leq 1}$$



$x = \sqrt{1-(y-2)^2}$

$$E = \{(x,y)^T \in \mathbb{R}^2 : \ y \in [1,3],\ 0 \leq x \leq \sqrt{1-(y-2)^2}\}$$

$$= \int_1^3 \left(\int_0^{\sqrt{1-(y-2)^2}} \frac{x}{y}\,dx \right) dy = \int_1^3 \frac{1}{y} \cdot \left[\frac{1}{2} x^2 \right]_0^{\sqrt{1-(y-2)^2}} dy = \int_1^3 \frac{1}{y} \cdot \frac{1}{2} \left(1-(y-2)^2\right) dy$$

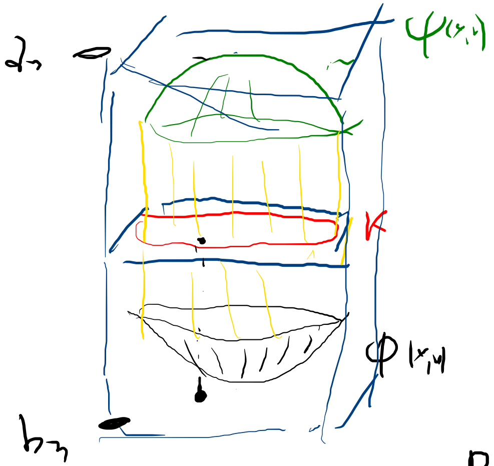
$1 - y^2 + 4y - 4 = -3 + \cancel{2}4y - y^2$

$$= \int_1^3 \frac{1}{2}\left(\frac{-3}{y} + \cancel{2}4 - y\right) dy = \left[-\frac{3}{2}\log y + 2y - \frac{1}{4}y^2 \right]_1^3 = -\frac{3}{2}\log 3 + 6 - \frac{1}{4}\cdot 9 - 2 + \frac{1}{4}$$

$$4 - 2 - \frac{3}{2}\log 3 = 2 - \frac{3\log 3}{2}$$

$E = \{(x,y,z)^T \in \mathbb{R}^3 : (x,y)^T \in K, \ \phi(x,y) \le z \le \psi(x,y)\}$ $\quad f : E \to \mathbb{R}$

$\Rightarrow$ f è integrabile su E e $\quad \iiint_E f\,dm = \iint_K \left(\int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz \right) dx\,dy$



Dim prendo R rettangolo $K \subseteq R$ e considero

$\phi_R^*, \psi_R^* : R \to \mathbb{R}$ $\qquad \phi_R^*(x,y) = \begin{cases} 0 & \text{se } (x,y)^T \in R \setminus K \\ \phi(x,y) & \text{se } (x,y)^T \in K \end{cases}$

ψ_R^*

considero $a_3 = \min_R \phi^*$ $\qquad b_3 = \max_R \psi^*$

Prendo $R \times [a_3, b_3] = Q$ $\qquad E \subset Q$

$$\iiint_E f\,dm = \iiint_Q f_Q^*\,dm = \iint_R \left(\int_{[a_3,b_3]} f^*(x,y,z)\,dz \right) dx\,dy =$$

Fubini per coroll

$f^* = 0$ se $(x,y)^T \in R \setminus K$

se $(x,y)^T \in K$ $= 0$ se $z < \phi(x,y)$ oppure $z > \psi(x,y)$

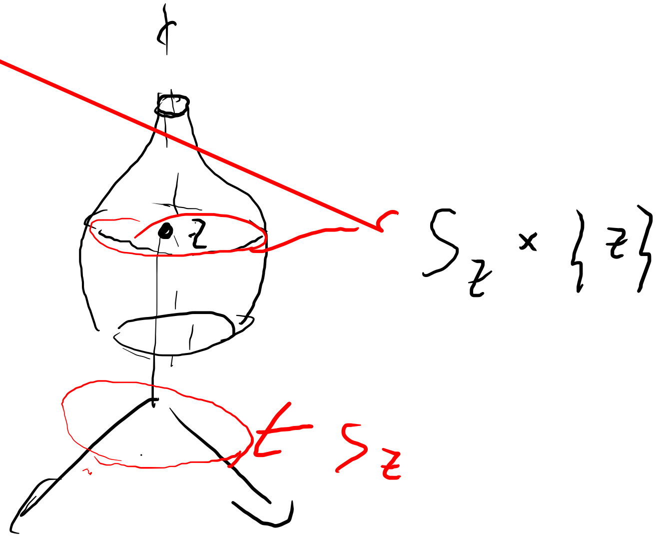
$= f(x,y,z)$ se $(x,y)^T \in K$ e $\phi(x,y) \le z \le \psi(x,y)$

$$= \iint_K \left(\int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz \right) dx\,dy.$$

Riduzione per sezioni

Sia $E \subseteq \mathbb{R}^3$, $\hat{z} \in \mathbb{R}$. Si dice sezione di quota $\hat{z}$ di E l'insieme

$$S_{\hat{z}} = \{ (x,y)^T \in \mathbb{R}^2 : (x, y, \hat{z})^T \in E \}$$



Sia $E \subseteq \mathbb{R}^3$ un insieme chiuso e $\boxed{\text{misurabile}}$. E si dice sezionabile rispetto all'asse z se tutte le sezioni di E sono misurabili in $\mathbb{R}^2$

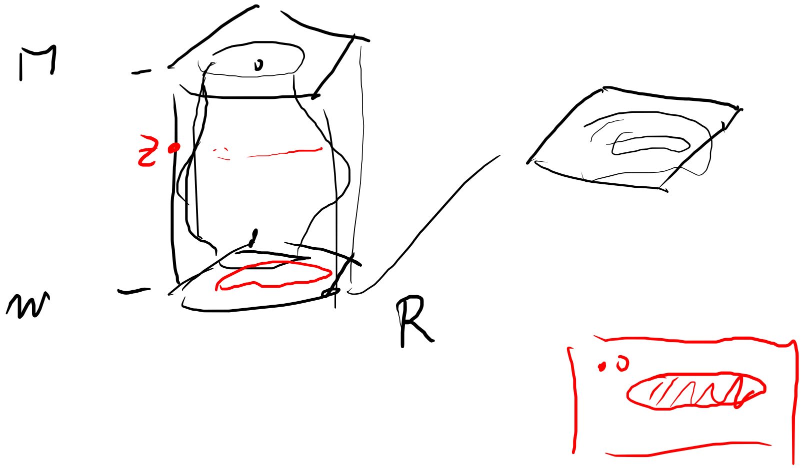
<u>Teorema</u> (Riduzione per sezioni)

Sia E sezionabile rispetto all'asse z. Sia $f: E \to \mathbb{R}$ continua. Allora f è integrabile su E e si ha

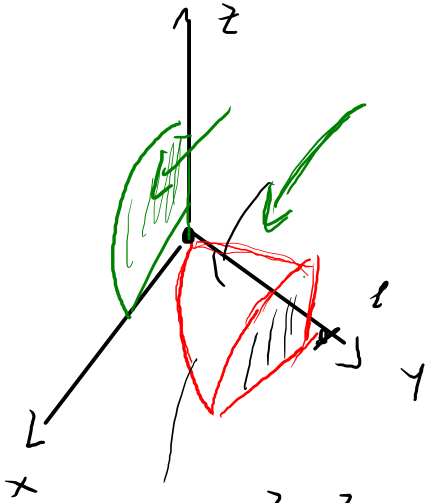
$$\iiint_E f\,dm = \int_m^M \left(\iint_{S_z} f(x,y,z)\,dx\,dy \right) dz$$

dove $m = \min\{ z \in \mathbb{R} : S_z \neq \emptyset \}$

$M = \max\{ \quad " \quad \}$

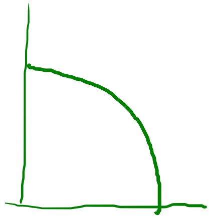


$$\iiint_E f\,dm = \iiint_E f^* \,dm = \int_m^M \left(\underbrace{\iint_R f^*(x,y,z)\,dx\,dy}_{\displaystyle \iint_{S_z} f(x,y,z)\,dx\,dy} \right) dz$$



Esempio

$$\iiint_E 2xz\,dx\,dy\,dz = *$$



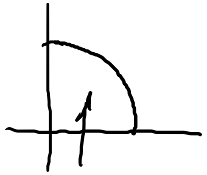
$z^2 = y - x^2$

$z^2 + x^2 = 1$

$z^2 = y - x^2$

E solido del primo ottante limitato dalle superfici $y = x^2+z^2$ e $y=1$

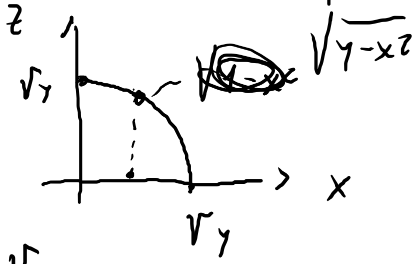
$y = x^2 + z^2$



S_y

$x^2+z^2 = \sqrt{y}^2$

$r = \sqrt{y}$



$S_y = \{(x,z)^T \in \mathbb{R}^2 :$

$0 \le x \le \sqrt{y},\ 0 \le z \le$

$$* = \int_0^1 \left(\iint_{S_y} 2xz\,dx\,dz \right) dy$$

$$\iint_{S_y} 2xz\,dx\,dz = \int_0^{\sqrt{y}} \left(\int_0^{\sqrt{y-x^2}} 2xz\,dz \right) dx = \circledast$$

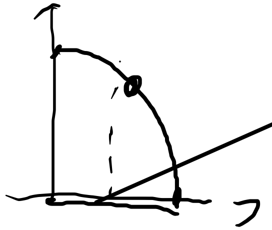
$$= \frac{1}{4} y^2$$

$$\circledast = \int_0^{\sqrt{y}} x\,(y - x^2)\,dx = \left[\frac{1}{2} x^2 y - \frac{1}{4} x^4 \right]_0^{\sqrt{y}} = \frac{1}{2} y^2 - \frac{1}{4} y^2 = \frac{1}{4} y^2$$

$xy - x^3$

$$= \int_0^1 \frac{1}{4} y^2\,dy = \frac{1}{12}$$

$$E = \{ (x,y,z)^T \in \mathbb{R}^3 : (x,z)^T \in K, \quad x^2+z^2 \le y \le 1 \}$$



$x \in [0,1]$

$0 \le z \le \sqrt{1-x^2}$

$$K = \{ (x,z)^T : x^2+z^2 \le 1 \}$$

$x \geq 0 \quad z \geq 0$

$$\iiint_E f \, dm = \iint_K \left(\int_{x^2+z^2}^{1} 2xz \, dy \right) dx\,dz = \iint_K 2xz\left(1-x^2-z^2\right) dx\,dz =$$

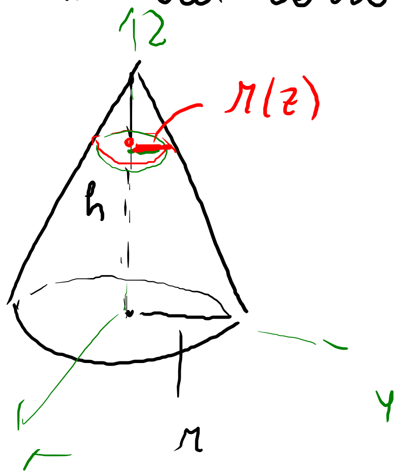
$$= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(2xz - 2x^3 z - 2xz^3\right) dz \right) dx = \int_0^1 \left(x(1-x^2) - x^3 \cdot (1-x^2) - \frac{1}{2} x (1-x^2)^2 \right) dx$$

$1-2x^2+x^4$

$x - x^3 \cancel{-x^3} + x^5 - \frac{1}{2}x + \cancel{x^3} - \frac{1}{2}x^5$

$$\int_0^1 \left(\frac{1}{2}x - x^3 + \frac{1}{2}x^5\right) dx = \frac{1}{4} - \frac{1}{4} + \frac{1}{12} = \frac{1}{12}$$

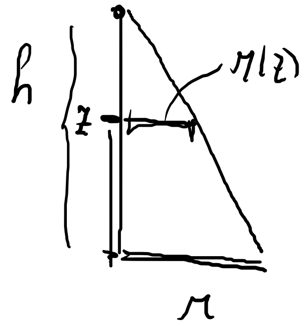
Volume del cono



$$V = \iiint_E 1 \, dm = \int_0^h \Big(\underbrace{\iint_{S_z} dx\,dy}_{} \Big) dz = *$$

$$\text{Area di } S_z = \pi \, r(z)^2 = \pi \left(\frac{h-z}{h} r \right)^2$$

come calcolare $r(z)$?



$$\frac{h}{r} = \frac{h-z}{r(z)} \qquad r(z) = \frac{h-z}{h} r$$

$$* = \int_0^h \pi \left(1 - \frac{z}{h}\right)^2 r^2 \, dz = \pi r^2 \cdot \left[-h \left(1 - \frac{z}{h}\right)^3 \cdot \frac{1}{3} \right]_0^h = \pi r^2 \cdot h \cdot \frac{1}{3} \cdot$$

Cambio di variabili negli integrali in $\mathbb{R}^n$

$n=1$ $\quad \displaystyle\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt \qquad \varphi(\alpha)=a \quad \varphi(\beta)=b \qquad \varphi\in C^1$

Sia φ invertibile C^1 con inverso C^1 (φ è un C^1-diffeomorfismo) $\quad \varphi'(t)\neq 0 \ \forall t$

$\varphi'(t)>0 \ \forall t$, φ crescente $\qquad \varphi:[\alpha,\beta]\to[a,b]$

$\varphi(\alpha)=a \qquad \varphi(\beta)=b$

$$\int_{[a,b]} f(x)\,dx = \int_{[\alpha,\beta]} f(\varphi(t))\cdot\underbrace{|\varphi'(t)|}_{=\varphi'(t)}\,dt$$

$[a,b] = \varphi([\alpha,\beta])$

$\varphi'(t)<0 \ \forall t$ $\quad \varphi$ decrescente

$\varphi:[\alpha,\beta]\to[b,a] \qquad (\varphi(\alpha)=a>b=\varphi(\beta))$

$$\int_{[b,a]} f(x)\,dx = \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = -\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$$= \int_{[\alpha,\beta]} f(\varphi(t))\cdot\underbrace{|\varphi'(t)|}_{=-\varphi'(t)}\,dt$$

$[b,a] = \varphi([\alpha,\beta])$

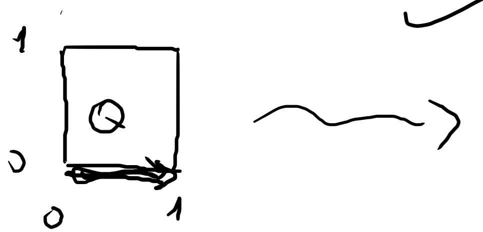
$$\boxed{\int_{\varphi(E)} f(x)\,dx = \int_E f(\varphi(t))\cdot|\varphi'(t)|\,dt}$$

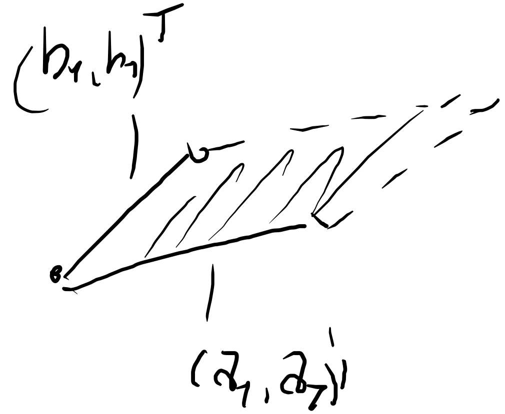
?

Es. $N=2$ $\quad \varphi(s,t) = \big(x(s,t),\, y(s,t)\big)^T \quad \varphi : \mathbb{R}^2 \to \mathbb{R}^2$

Esempio della trasformazione lineare di coordinate

$$\varphi\begin{pmatrix} s \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}_{A} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} a_1 s + b_1 t \\ a_2 s + b_2 t \end{pmatrix}$$



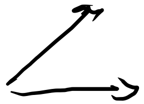


$Q = \{ (s,t)^T : 0 \le s \le 1,\ 0 \le t \le 1 \}$ $\qquad \varphi(Q) \overset{?}{=}$

quadrato $\qquad$ parallelogramma

Area del parallelogramma ? $= \det A$

$Q = [0,1]^2$ $\qquad \varphi(Q)$ parallelogramma individuato da $(a_1,a_2)^T$ e $(b_1,b_2)^T$



$$\text{Area}(Q) = \iint_Q 1 \, dm = 1$$

$$\text{Area}(\varphi(Q)) = \iint_{\varphi(Q)} 1 \, dx\,dy = \left|\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\right| = \iint_Q \left|\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_1 \end{pmatrix}\right| ds\,dt$$

$$|\det J\varphi|$$

$$\varphi \qquad J\varphi = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$$\Rightarrow \boxed{\iint_{\varphi(E)} f(x,y)\,dx\,dy = \iint_E f(\varphi(s,t)) \cdot |\det J\varphi(s,t)| \, ds\,dt}$$

<u>Teorema</u>

Sia $A, B \subseteq \mathbb{R}^n$ aperti misurabili; $\Phi : B \to A$ C^1-diffeomorfismo

[cioè Φ è invertibile C^1, Φ^{-1} è C^1; $\det J\Phi(u,v) \neq 0 \ \forall (u,v)^T \in B$]

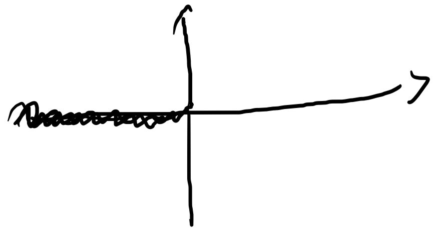
$f : A \to \mathbb{R}$ continua e limitata. Allora

$$\iint_B f(x,y)\,dx\,dy = \iint_A f(\Phi(u,v)) \cdot \overbrace{|\det J\Phi(u,v)|\,du\,dv}^{\text{elemento di area}}$$

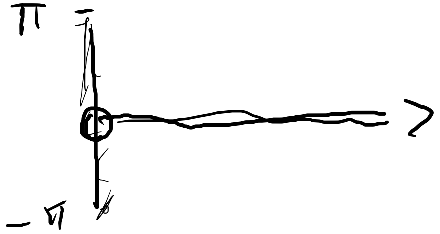
$$\iiint_B f(x,y,z)\,dx\,dy\,dz = \iiint_A f(\Phi(u,v,w)) \cdot \underbrace{|\det J\Phi(u,v,w)|\,du\,dv\,dw}_{\text{elemento di volume}}$$

Esempio: coordinate polari

$A = \mathbb{R}^2 - \{ (x,0)^T : x \le 0 \}$



$B = \underbrace{]0,+\infty[}\times \underbrace{]-\pi,\pi[}$ ($]0,2\pi[$



$\phi : B \to A \qquad \phi(\rho,\vartheta) = (\rho\cos\vartheta,\ \rho\,\mathrm{sen}\,\vartheta)^T$

⚠ la ϕ definita su $[0,+\infty[\times[-\pi,\pi]$ non è biiettiva

$$J\phi(\rho,\vartheta) = \begin{pmatrix} \cos\vartheta & -\rho\,\mathrm{sen}\,\vartheta \\ \mathrm{sen}\,\vartheta & \rho\cos\vartheta \end{pmatrix} \qquad |\det J\phi(\rho,\vartheta)| = \rho > 0$$

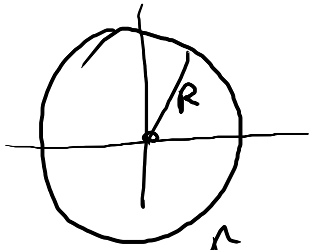
$$\iint_{E_{xy}} f(x,y)\,dx\,dy = \iint_{E_{\rho\theta}} f(\rho\cos\vartheta,\ \rho\,\mathrm{sen}\,\vartheta)\cdot \boxed{\rho}\ d\rho\, d\vartheta$$

!!!

Coordinate ellittiche

$$\begin{cases} x = a\rho\cos\vartheta \\ y = b\rho\,\mathrm{sen}\,\vartheta \end{cases} \qquad \rho \le 1 \qquad |\det J\varphi| = ab\rho$$

Area del cerchio



$E_{xy} = \{(x,y)^T : x^2+y^2 \le R^2\}$

$E_{\rho\vartheta} = \{(\rho,\vartheta)^T : 0 \le \rho \le R,\ \vartheta \in [0, 2\pi]\}$

$$\iint_{E_{xy}} 1\,dx\,dy = \iint_{E_{\rho\vartheta}} \rho\,d\rho\,d\vartheta = \int_0^R \left(\int_0^{2\pi} \rho\,d\vartheta\right) d\rho = \int_0^R \rho \cdot 2\pi\,d\rho = \pi R^2$$