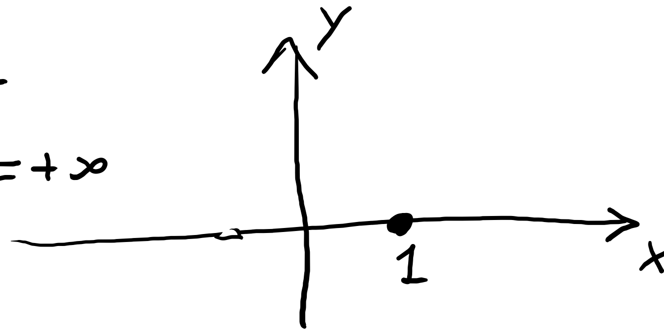
20 Novembre

Studi di funzione.

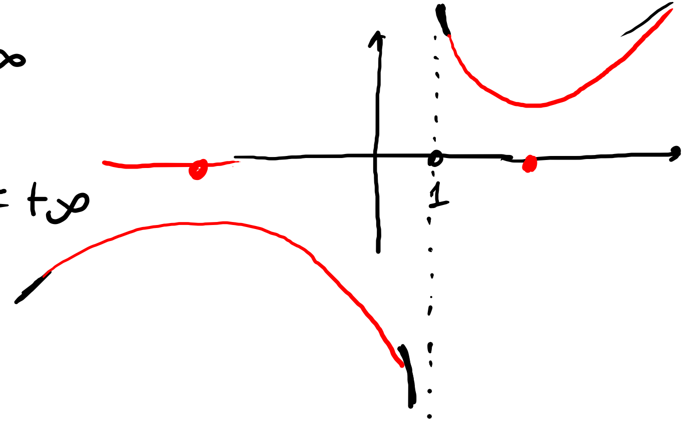
Studiare il grafico di $f(x) = \dfrac{x^2+1}{x-1}$

1) Dominio $\quad x \neq 1$



2) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \dfrac{x^2}{x} = +\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x = -\infty$$



$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{x^2+1}{x-1} = +\infty$$

$$\lim_{x\to1^-} \frac{x^2+1}{x-1} = -\infty$$

3) $f(x) > 0$ per $x > 1$

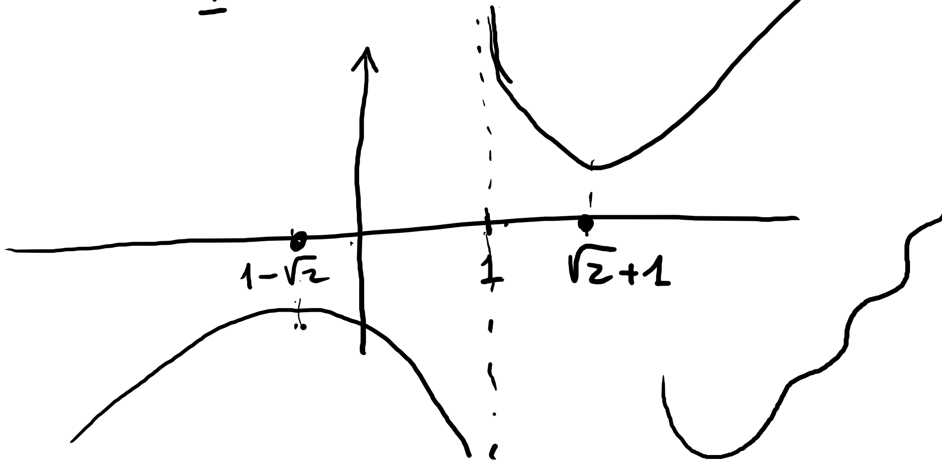
$ < 0 \qquad x < 1$

4) $f'(x) = \left(\dfrac{x^2+1}{x-1}\right)' = \dfrac{2x(x-1)-(x^2+1)}{(x-1)^2} = \dfrac{2x^2-2x-x^2-1}{(x-1)^2}$

$= \dfrac{x^2-2x-1}{(x-1)^2} = 0$ $\quad x_{\pm} = 1 \pm \sqrt{2}$

$f'(x) > 0$ per $x \notin [-\sqrt{2}+1, \sqrt{2}+1]$

$f'(x) < 0$ per $x \in (-\sqrt{2}+1, \sqrt{2}+1)$

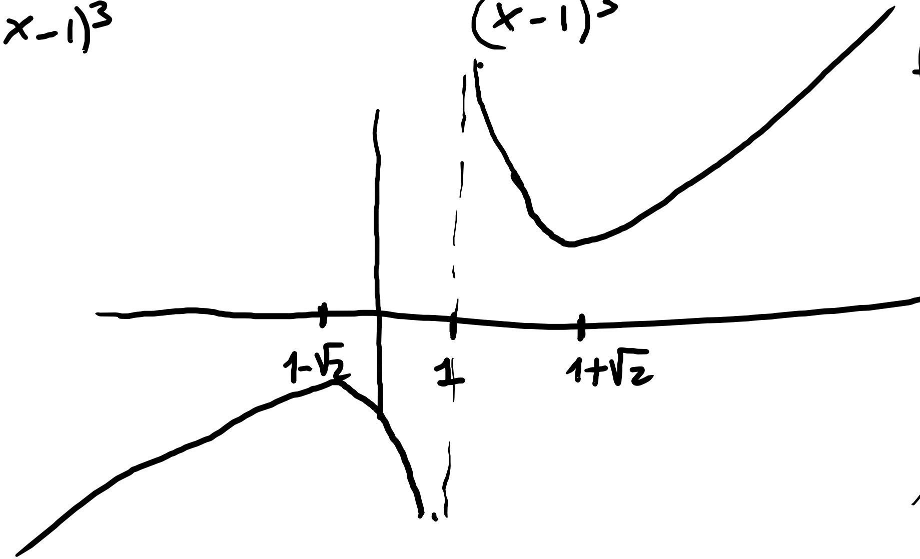


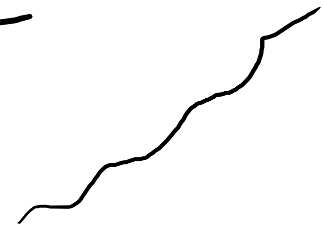
$$f' = \frac{x^2-2x-1}{(x-1)^2} \qquad f''(x) = \frac{(2x-2)(x-1)^{\cancel{2}1} - (x^2-2x-1)\,2\cancel{(x-1)}}{(x-1)^{\cancel{4}3}}$$

$$= 2\,\frac{\cancel{x^2}\cancel{-2x}+1-\cancel{x^2}\cancel{+2x}+1}{(x-1)^3} = \frac{4}{(x-1)^3}$$

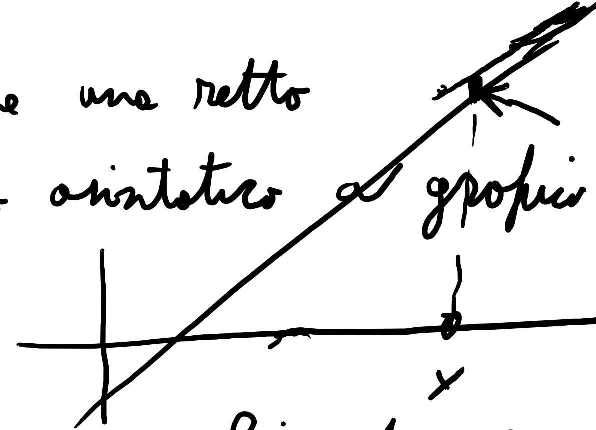
$f'' > 0$ når $x > 1$

$f'' < 0$ når $x < 1$





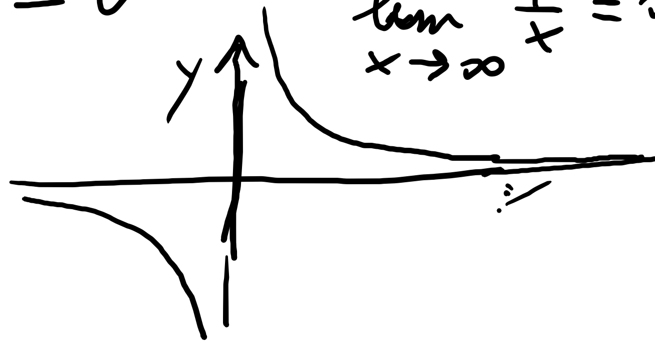
Dato $f: \mathbb{R} \to \mathbb{R}$ si dice che una retta $y = mx + c$ è la retta asintotica al grafico $y = f(x)$ se



$$\lim_{x \to +\infty} \left[f(x) - (mx + c) \right] = 0$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Es $f(x) = \frac{1}{x}$



$y = 0$ è asintotica sia a $+\infty$ che a $-\infty$.

Come si trova una retta asintotica. Si cerca una retta $y = mx + c$ t.c.

$$\lim_{x \to +\infty} [f(x) - mx - c] = 0 \Rightarrow \lim_{x \to +\infty} \frac{f(x) - mx - c}{x} = 0$$

$$\Longleftrightarrow \lim_{x \to +\infty} \left(\frac{f(x)}{x} - m \right) = 0 \Longleftrightarrow \lim_{x \to +\infty} \frac{f(x)}{x} = m_0$$

$y = m_0 x + c$

$$\lim_{x \to +\infty} [f(x) - m_0 x - c] = 0 \Longleftrightarrow \lim_{x \to +\infty} [f(x) - m_0 x] = c_0$$

$y = m_0 x + c_0$

$f \quad f(x) = \dfrac{x^2+1}{x-1}$

$\lim_{x \to +\infty} \dfrac{f(x)}{x} = ?$

$\lim_{\substack{x \to +\infty \\ -\infty}} \dfrac{x^2+1}{(x-1)x} = \lim_{\substack{x \to +\infty \\ -\infty}} \dfrac{x^2}{x^2} = 1 = m$

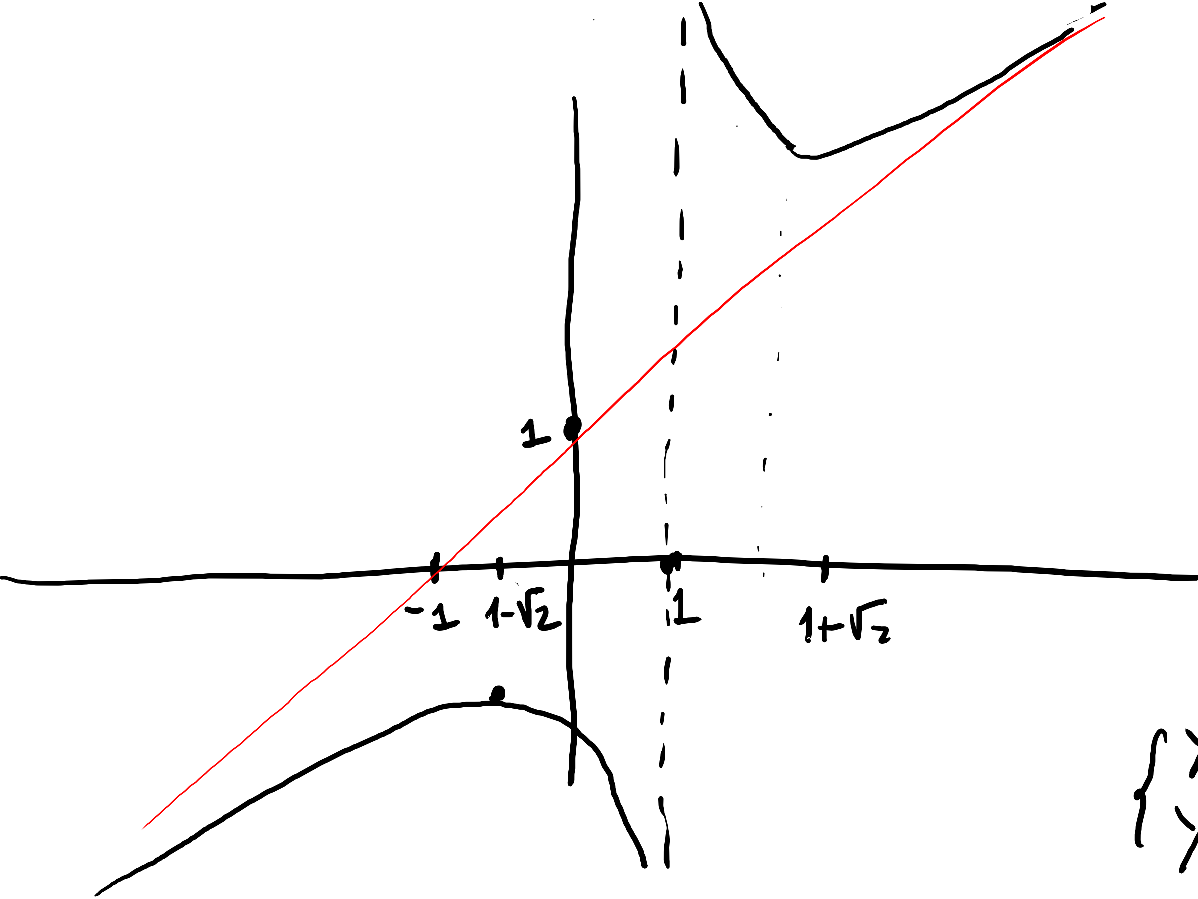
$y = x + 1$

$\lim_{x \to +\infty} \left[f(x) - mx \right]$

$\lim_{x \to +\infty} \left[\dfrac{x^2+1}{x-1} - x \right] = \lim_{x \to +\infty} \left[\dfrac{x^2+1-x(x-1)}{x-1} \right]$

$= \lim_{x \to +\infty} \dfrac{1+x}{x-1} = \lim_{x \to +\infty} \dfrac{x}{x} = 1$

$y = x + 1$



$-1 < 1-\sqrt{2}$

$\sqrt{2} < 2$

$2 < 4$

$$\begin{cases} y = 0 \\ y = x + 1 \end{cases}$$

$$f(x) = (x^2 + x + 1)e^{-x}$$

1) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{x^2 + x + 1}{e^x} = 0$

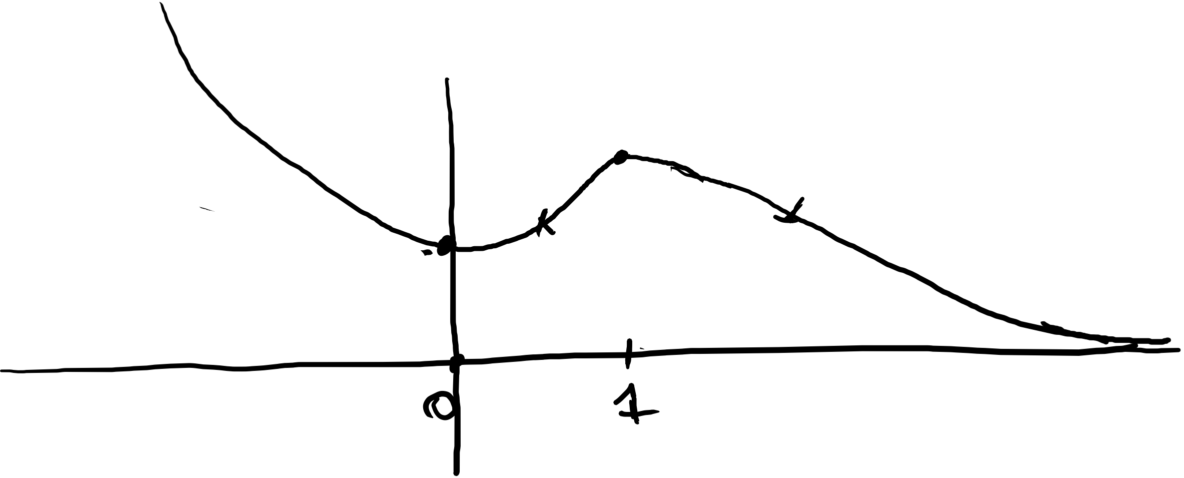
$\lim\limits_{x \to -\infty} f(x) = +\infty$

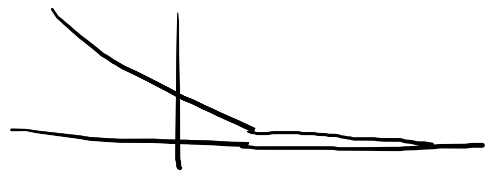
2) $f(x) > 0 \quad \forall x$

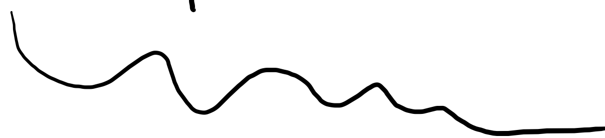
$x^2 + x + 1 = 0$

$x_\pm = -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$

$\Rightarrow \quad x^2 + x + 1 > 0 \qquad \forall x \in \mathbb{R}$


0
1

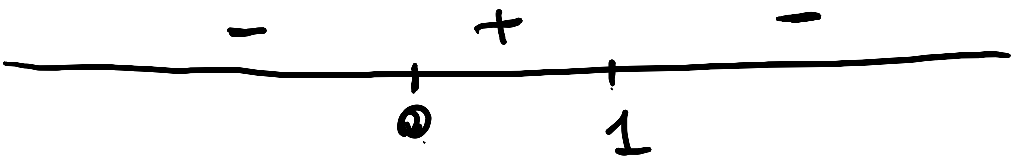


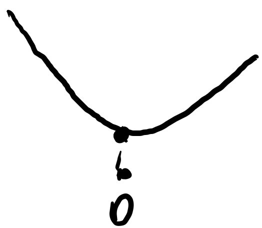


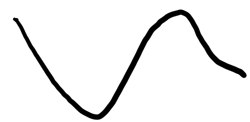
$$f' = (2x + \cancel{1})\, e^{-x} - (x^2 + x + \cancel{1})\, e^{-x} =$$

$$= (x - x^2)\, e^{-x} = x(1-x)\, e^{-x}$$

$f'(x) = 0$ per $x = 0$ $\quad x = 1$







$$f'(x) = (x - x^2)\, e^{-x}$$

$$f''(x) = \big((1-2x) - (x-x^2)\big)\, e^{-x}$$

$$f''(x) = (1 - 3x + x^2)\, e^{-x}$$

$$x^2 - 3x + 1 = 0$$

$$x_\pm = \frac{3}{2} \pm \frac{\sqrt{9-4}}{2} =$$

$$= \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

$$\boxed{0 < \frac{3}{2} - \frac{\sqrt{5}}{2} < 1}$$

$$\frac{3}{2} + \frac{\sqrt{5}}{2} > 1 \quad \checkmark$$

$$3 - \sqrt{5} < 2 \qquad 1 < \sqrt{5}$$

Lemma Consideriamo $x^a : \mathbb{R}_+ \to \mathbb{R}$, $a \in \mathbb{R}$

$$(x^a)^{(n)} = \prod_{j=1}^{n} (a-j+1)\, x^{a-n} \qquad *$$

Corollario se $a = m \in \mathbb{N}$ $*$ implica

che $(x^m)^{(n)} = 0$ per $n > m$

$$(x^m)^{(n)} = \prod_{j=1}^{n} (m-j+1) \quad x^{m-n} \qquad m \in \mathbb{N}, \; n > m$$

$$= m(m-1)\cdot\ldots\cdot\underbrace{(m-n+1)}_{\leq 0} x^{m-n}$$

$n > m \Rightarrow n \geq m+1$

Se $n = m+1$ allora $m-n+1 = 0$

se $n > m+1$ allora uno dei fattori alla sinistra di $m-n+1$ è $=0$.

$$(x^a)^{(n)} = \prod_{j=1}^{n} (a-j+1)\ x^{a-n}$$

1) $n=1$ $\qquad (x^a)' = a\ x^{a-1} = \prod_{j=1}^{1} (a-j+1)\ x^{a-1}$

2) $n-1$

$$(x^a)^{(n)} = \left((x^a)^{(n-1)} \right)'$$

$$(x^a)^{(n)} = \left((x^a)^{(n-1)} \right)' = \left(\prod_{j=1}^{n-1} (a-j+1)\, x^{a-(n-1)} \right)' =$$

$$= \underbrace{\prod_{j=1}^{n-1} (a-j+1)}_{} \quad \underbrace{(a-n+1)}_{} \quad x^{a-n}$$

$$= \prod_{j=1}^{n} (a-j+1)\, x^{a-n} \qquad \square$$