Teorema di Fubini

f integrabile su $R = [a,b]\times[c,d]$ $\quad \forall x$ sia integrabile su $[c,d]$ la funzione $f(x,\cdot)$

$\Rightarrow \quad g(x) = \int_c^d f(x,y)\,dy$ è integrabile su $[a,b]$ e $\int_a^b g(x)\,dx = \iint_R f\,dm$.

Per mostrare che g è integrabile si deve verificare che $\forall \varepsilon>0$ esiste una decomposizione $\mathcal{S}_x = \{[x_{i-1},x_i] : i=1,\dots,N\}$ di $[a,b]$ tale che $S(g,\mathcal{S}_x) - s(g,\mathcal{S}_x) < \varepsilon$.

Proveremo che se $\mathcal{S} = \{R_{ij} \quad i=1\dots N,\ j=1\dots M\}$ è una decomposizione di R $\quad R_{ij} = I_i\times J_j$

$I_i = [x_{i-1},x_i] \quad J_j = [y_{j-1},y_j]$ allora, posto $\mathcal{S}_x = \{I_i : i=1\dots N\}$ $\quad \mathcal{S}_y = \{J_j : j=1\dots M\}$

$$S(g,\mathcal{S}_x) - s(g,\mathcal{S}_x) \le S(f,\mathcal{S}) - s(f,\mathcal{S})$$

L'osservazione importante è che
$(\forall\ h_1,h_2 : I\to\mathbb{R})$

$$\boxed{\begin{aligned}\sup_{x\in I}\big(h_1(x)+h_2(x)\big) &\le \sup_{x\in I} h_1(x) + \sup_{x\in I} h_2(x)\\ \inf_{x\in I}\big(h_1(x)+h_2(x)\big) &\ge \inf_I h_1 + \inf_I h_2\end{aligned}}$$

$(*)$

Sappiamo che $\boxed{\forall x}$ $g(x,\cdot)$ è integrabile

$$s(f(x,y), \delta_y) \le \underbrace{\int_c^d f(x,y)\,dy}_{=\,g(x)} \le S(f(x,y), \delta_y)$$

nella variabile y

$\Downarrow$

$x \in I_i$

$$\sup_{x\in I_i} g(x) \le \sup_{x\in I_i}\left[\sum_{j=1}^{M} \sup_{y\in J_j} f(x,y)\cdot m(J_j)\right] \le \sum_{j=1}^{M} \sup_{(x,y)^T\in R_{ij}} f(x,y)\cdot m(J_j)$$

moltiplico per $m(I_i)$

$$S(g, \delta_x) \le S(f, \delta)$$

in modo simile provo che $s(g, \delta_x) \ge s(f, \delta)$.

Massa, centro di massa   Momento d'inerzia

Volume E $= \int_E dm$ sia E un oggetto con una densità di massa $\delta(x)$

Massa $= \int_E \delta(x)\, dm$ $\mathbb{R}^2$ $\iint_E \delta(x,y)\, dx\, dy$ = Massa

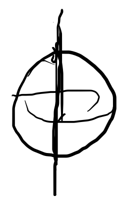
$\mathbb{R}^3$ $\iiint_E \delta(x,y,z)\, dx\, dy\, dz$

Centro di massa $(\hat{x},\hat{y},\hat{z})^T$

$$\hat{x} = \frac{1}{M} \iiint_E x\, \delta(x,y,z)\, dx\, dy\, dz \qquad \hat{y} = \frac{1}{M} \iiint_E y\, \delta(x,y,z)\, dx\, dy\, dz$$

$$\hat{z} = \frac{1}{M} \dots$$

se δ è costante si può supporre $\delta = 1$.



Momento d'inerzia rispetto ad un asse

$$\boxed{I_z = \iiint_E (x^2+y^2)\, \delta(x,y,z)\, dx\, dy\, dz}$$

Momento di inerzia di un solido E rispetto ad un insieme C

$$I_C = \iiint_E d^2\left((x,y,z)^T, C\right)\, \delta(x,y,z)\, dx\, dy\, dz$$

assi $\quad I_z \qquad I_x = \iiint_E (y^2+z^2)\, \delta(x,y,z)\, dx\, dy\, dz$

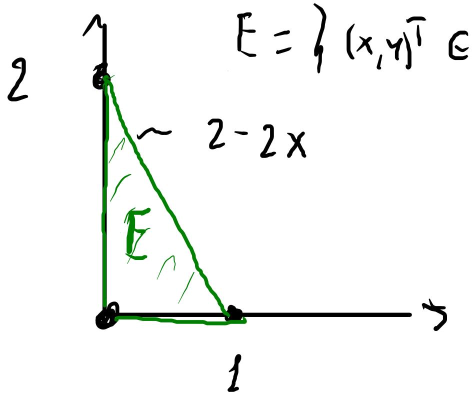
$$I_y = \cdots (x^2+z^2) \cdots$$

piano $\quad I_{xy} = \iiint_E z^2\, \delta(x,y,z)\, dx\, dy\, dz$

punto $\quad I_{(0,0,0)^T} = \iiint_E (x^2+y^2+z^2)\, dx\, dy\, dz$

$(0,0)^T \qquad (1,0)^T \qquad (0,2)^T \qquad\qquad \delta(x,y) = 1+3x+y$

$E = \{(x,y)^T \in \mathbb{R}^2 : \; x \in [0,1], \; 0 \le 2-2x$



$\hat{x} = \frac{3}{8}$.

~~Massa~~ $= \iint_E x\,\delta(x,y)\,dx\,dy = \iint_E x(1+3x+y)\,dx\,dy = \dots$

$$= \int_0^1 \left(\int_0^{2-2x} (1+3x+y)\,dy \right) dx = \int_0^1 \left[(1+3x)\,2(1-x) + \frac{1}{\cancel{2}} \left(\cancel{4}^{\,2} (1-x)^2 \right) \right] dx = 2 \int_0^1 (2-2x^2)\,dx =$$

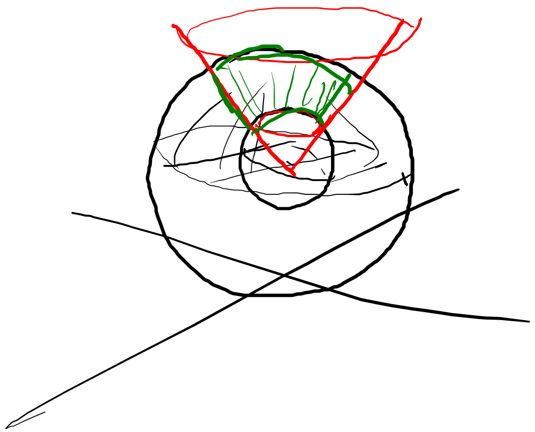
$\frac{1}{2}y^2$

$1+\cancel{3x}-\cancel{x}-3x^2 + 1 + x^2 - \cancel{2x}$

$$= 4\int_0^1 (1-x^2)\,dx = 4\cdot\left(1-\frac{1}{3}\right) = \boxed{8/3} \qquad \text{massa} = 8/3$$

$$I_{xy} = \iiint_S z^2 \, dm$$

$$1 \leq x^2+y^2+z^2 \leq 4$$

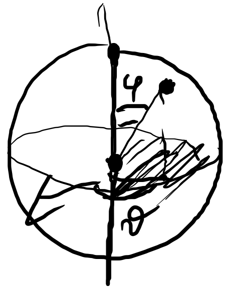


$$z \geq 0$$

$$z^2 \geq x^2+y^2$$

$$|z| \geq \sqrt{x^2+y^2}$$

$$z \geq 0 \qquad z = \sqrt{x^2+y^2}$$



Coordinate sferiche

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\vartheta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \\ z = \rho \cos\varphi \end{cases}$$

$$\Phi :]0,+\infty[\times]0,\pi[\times]0,2\pi[\longrightarrow \mathbb{R}^3 \setminus \text{qualcosa}$$

$\rho > 0$ $\qquad$ $\varphi \neq 0 \;\; \varphi \neq \pi$ $\qquad$ $\vartheta \neq 0 \;\; 2\pi$

$$J_\varphi = \begin{pmatrix} \operatorname{sen}\varphi \cos\vartheta & \rho \cos\varphi \cos\vartheta & -\rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \\ \operatorname{sen}\varphi \operatorname{sen}\vartheta & \rho \cos\varphi \operatorname{sen}\vartheta & \rho \operatorname{sen}\varphi \cos\vartheta \\ \cos\varphi & -\rho \operatorname{sen}\varphi & 0 \end{pmatrix}$$

$x = \rho \operatorname{sen}\varphi \cos\vartheta$

$y = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta$

$z = \rho \cos\varphi$

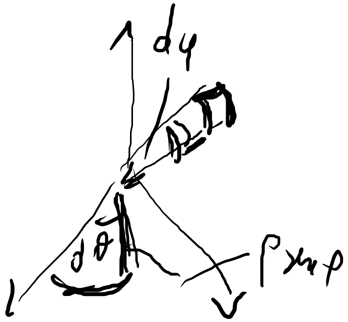
$$\boxed{\det J_\varphi} = +\cos\varphi \cdot \left(\rho^2 \operatorname{sen}\varphi \cos\varphi \cos^2\vartheta + \rho^2 \operatorname{sen}\varphi \cos\varphi \operatorname{sen}^2\vartheta \right) +$$

$$- (-\rho \operatorname{sen}\varphi) \cdot \left(\rho \operatorname{sen}^2\varphi \cos^2\vartheta + \rho \operatorname{sen}^2\varphi \operatorname{sen}^2\vartheta \right)$$

$$= \cos\varphi \cdot \rho^2 \operatorname{sen}\varphi \cos\varphi + \rho \operatorname{sen}\varphi \cdot \rho \operatorname{sen}^2\varphi =$$

$$= \boxed{\rho^2 \operatorname{sen}\varphi}$$

elemento di volume

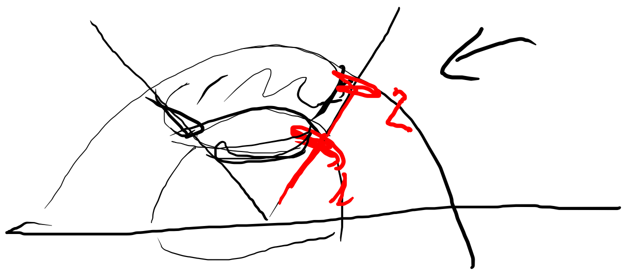


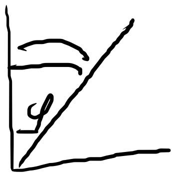
$$\iiint_{E_{xyz}} f(x,y,z)\, dx\, dy\, dz = \iiint_{E_{\rho\varphi\vartheta}} f(\rho \operatorname{sen}\varphi \cos\vartheta, \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta, \rho\cos\varphi) \cdot \boxed{\rho^2 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\vartheta}$$

$d\rho \cdot \rho \operatorname{sen}\varphi\, d\vartheta \cdot$

$\cdot\, \rho\, d\varphi$

$\boxed{1 \le x^2+y^2+z^2 \le 4}$ $\qquad z \ge 0 \qquad z^2 \ge x^2+y^2$





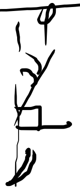
$x = \rho \operatorname{sen}\varphi \cos\vartheta$

$y = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta$

$z = \rho \cos\varphi$

$1 \le \rho^2 \le 4 \qquad \rho \in [1,2]$

$z \ge 0 \qquad \rho\cos\varphi \ge 0 \Rightarrow \varphi \in [0, \frac{\pi}{2}]$



$\rightsquigarrow \varphi \in [0, \frac{\pi}{4}]$

$z^2 \ge x^2+y^2 \qquad \rho^2\cos^2\varphi \ge \rho^2 \operatorname{sen}^2\varphi \cos^2\vartheta + \rho^2 \operatorname{sen}^2\varphi \operatorname{sen}^2\vartheta = \rho^2 \operatorname{sen}^2\varphi$

$\rho > 0 \qquad \cos^2\varphi \ge \operatorname{sen}^2\varphi \Rightarrow \operatorname{tg}^2\varphi \le 1 \rightsquigarrow \boxed{\varphi \le \frac{\pi}{4}}$

$\varphi \in [0, \frac{\pi}{2}]$

condizioni su ϑ?

NO $\qquad \vartheta \in]0, 2\pi[$

$$\int z^2\,dm = \iiint_{[1,2]\times[0,\frac{\pi}{4}]\times[0,2\pi]} \rho^2\cos^2\varphi \cdot \rho^2 \operatorname{sen}\varphi \; d\rho\, d\varphi\, d\vartheta =$$

$\rho \in [1,2]$ $\quad$ $\vartheta \in [0,2\pi]$ $\quad$ $\varphi \in [0,\frac{\pi}{4}]$

$z = \rho\cos\varphi$

$$= \int_1^2 \left(\iint_{[0,\frac{\pi}{4}]\times[0,2\pi]} \rho^4 \operatorname{sen}\varphi \cos^2\varphi \; d\varphi\, d\vartheta \right) d\rho = \underbrace{\int_1^2 \rho^4\, d\rho}_{\frac{1}{5}(2^5-1)} \cdot \int_0^{\pi/4} \left(\int_0^{2\pi} \operatorname{sen}\varphi \cos^2\varphi\, d\vartheta \right) d\varphi$$

$$= \frac{1}{5}\left(2^5-1\right) \cdot 2\pi \int_0^{\pi/4} \operatorname{sen}\varphi \cos^2\varphi\, d\varphi$$

$\cos\varphi = u \qquad du = -\operatorname{sen}\varphi\, d\varphi$

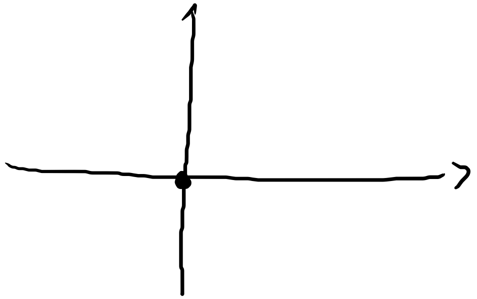
$$\int_1^{1/\sqrt{2}} u^2 (-1)\, du = \frac{1}{3}\left(1 - \frac{1}{2\sqrt{2}}\right)$$

$$\frac{\;\frac{2\sqrt{2}-1}{2\sqrt{2}}\;}{} \qquad \frac{2\cdot 2-\sqrt{2}}{4}$$

$$= \frac{2\pi}{15} \cdot (31) \cdot \frac{4-\sqrt{2}}{4} = \frac{31\pi}{30}(4-\sqrt{2})$$

Si calcoli il volume del solido ottenuto come intersezione del cono $x^2+y^2\le z^2$ e della sfera $x^2+y^2+z^2\le 2az$, con $a\ge 0$.

Sul piano xy

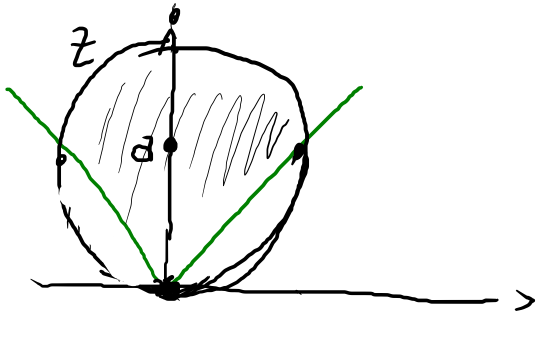


$z=0$ $\quad x^2+y^2\le 0$

$$\boxed{x^2+(z-a)^2\le a^2}$$

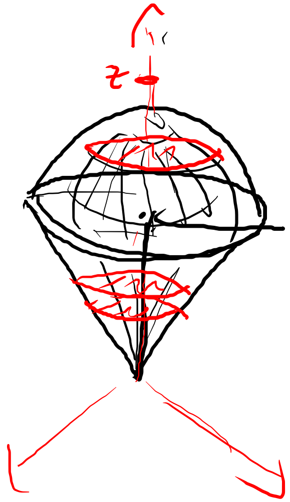
$$x^2+z^2-2az+a^2\le a^2$$

Sul piano xz



$y=0$ $\quad x^2+z^2\le 2az$

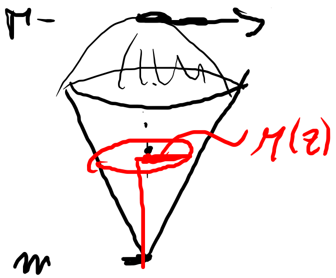
$x^2\le z^2$ — $|x|\le z$ $\quad z\ge 0$



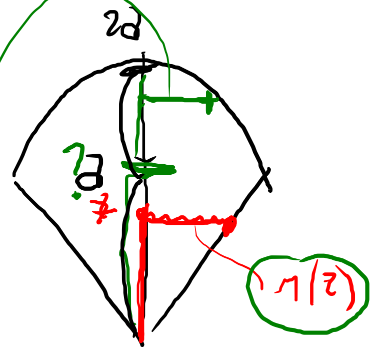
Volume della semisfera di raggio a $\quad \frac{1}{2}\,\frac{4}{3}\pi a^3$

del cono di $h=a$ e $r=a$ $\quad \frac{1}{3}\pi a^3$ $\quad\rightsquigarrow\quad \pi a^3$

Per sezioni



$$V_d = \iiint_E dm = \int_0^{2a} \left(\iint_{S_z} dx\,dy \right) dz = *$$



$x^2+y^2+z^2 \le 2az$ — $x=0,\ y=0 \Rightarrow z^2 = 2az$ $\quad z=2a$

$x^2+y^2 \le z^2$

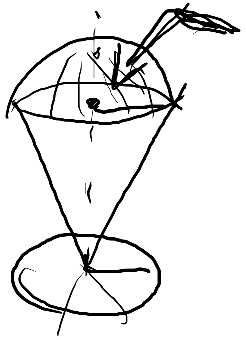
$r(z) = \sqrt{x^2+y^2} = z$ nella parte con il cono

$r(z) = \sqrt{x^2+y^2} = \sqrt{2az-z^2}$

$$\begin{cases} x^2+y^2 = z^2 \\ x^2+y^2+z^2 = 2az \end{cases} \Rightarrow 2z^2 = 2az \Rightarrow z = a$$

$$* = \int_0^{?=a} \pi \cdot z^2\, dz + \int_{?=a}^{2a} \pi\left(2az - z^2\right) dz = \pi a^3$$

Per coordinate



$$E = \{ (x,y,z)^T \in \mathbb{R}^3 : (x,y)^T \in K , \ \varphi(x,y) \le z \le \psi(x,y) \}$$

$$\underline{x^2+y^2 \le z^2}$$

$$x^2+y^2+z^2 \le 2az$$

$$\varphi(x,y) = \sqrt{x^2+y^2}$$

$$\psi(x,y) = a + \sqrt{a^2-x^2-y^2}$$

K ?

$$\| $$

$$B((0,0)^T, a)$$

$$\{ (x,y)^T \in \mathbb{R}^2 : x^2+y^2 \le a^2 \}$$

$$x^2+y^2+(z-a)^2 = a^2$$

$$(z-a)^2 = a^2-x^2-y^2$$

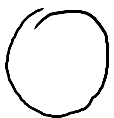
$z > a$

$$|z-a| = \sqrt{a^2-x^2-y^2}$$

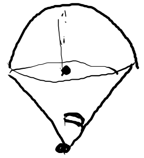
$$z = a + \sqrt{a^2-x^2-y^2}$$

$$\text{Vol.} = \iint_{\{(x,y)^T \in \mathbb{R}^2 : x^2+y^2 \le a^2\}} \left(\int_{\sqrt{x^2+y^2}}^{a+\sqrt{a^2-x^2-y^2}} dz \right) dx\,dy = *$$

$$* = \iint_{\{x^2+y^2 \le a^2\}} \left(a + \sqrt{a^2-x^2-y^2} - \sqrt{x^2+y^2} \right) dx\,dy = \int_0^{2\pi} \left(\int_0^a \left(a + \sqrt{a - \rho^2} - \rho \right) \rho \, d\rho \right) d\vartheta = \pi a^3$$



Coordinate sferiche



$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\vartheta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \\ z = \rho \cos\varphi \end{cases}$$

$x^2+y^2 \le z^2$ $\qquad$ $x^2+y^2+z^2 \le 2a z$

$\cancel{\rho^2} \operatorname{sen}^2\varphi \le \cancel{\rho^2} \cos^2\varphi$

$z \ge 0$

$\operatorname{tg}^2\varphi \le 1$ $\qquad$ $\varphi \in [0, \frac{\pi}{2}]$

$$\boxed{\varphi \in [0, \tfrac{\pi}{4}]}$$

$\vartheta \in [0, 2\pi]$

$$x^2+y^2 = \rho^2 \operatorname{sen}^2\varphi \underbrace{\cos^2\vartheta} + \rho^2 \operatorname{sen}^2\varphi \underbrace{\operatorname{sen}^2\vartheta} = \rho^2 \operatorname{sen}^2\varphi$$

$$\rho^2 \operatorname{sen}^2\varphi + \rho^2 \cos^2\varphi \le 2a\rho\cos\varphi$$

$$\Rightarrow \boxed{\rho \le 2a\cos\varphi}$$

$0 \le \rho \le$ funzione di φ

$$\iiint_E 1\, dm = \iiint_{\{(\rho,\varphi,\vartheta)^T:\ \varphi\in[0,\frac{\pi}{4}],\ \vartheta\in[0,2\pi],\ 0\le\rho\le 2a\cos\varphi\}} \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\vartheta =$$

$$= \iint_{\underset{\varphi}{[0,\frac{\pi}{4}]}\times\underset{\vartheta}{[0,2\pi]}} \left(\int_0^{2a\cos\varphi} \rho^2 \operatorname{sen}\varphi \, d\rho \right) d\varphi\, d\vartheta = \int_0^{\pi/4} \left(\int_0^{2\pi} \operatorname{sen}\varphi \cdot \frac{1}{3} 8a^3 \cos^3\varphi \, d\vartheta \right) d\varphi$$

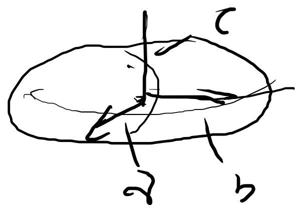
$$\left[\frac{1}{3}\rho^3\right]_0^{2a\cos\varphi}$$

$$= \frac{8}{3}a^3 \cdot 2\pi \int_0^{\pi/4} \operatorname{sen}\varphi \cdot \cos^3\varphi \, d\varphi$$

$= \pi a^3$

Coordinate ellissoidali

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



$$\begin{cases} x = a\rho \operatorname{sen}\varphi \cos\vartheta \\ y = b\rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \\ z = c\rho\cos\varphi \end{cases}$$

$$\begin{pmatrix} a\operatorname{sen}\varphi\cos\vartheta & a\rho\cos\varphi\cos\vartheta & -a\rho\operatorname{sen}\varphi\operatorname{sen}\vartheta \\ b\operatorname{sen}\varphi\operatorname{sen}\vartheta & b\rho\cos\varphi\operatorname{sen}\vartheta & b\rho\operatorname{sen}\varphi\cos\vartheta \\ c\cos\varphi & -c\rho\operatorname{sen}\varphi & 0 \end{pmatrix}$$

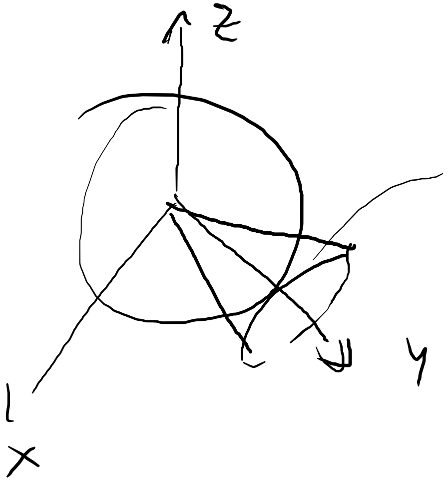
$$c\cos\varphi\,(a\rho\, b\rho\cos\varphi\operatorname{sen}\varphi) + c\rho\operatorname{sen}\varphi\,(a\, b\rho\operatorname{sen}^2\varphi) =$$

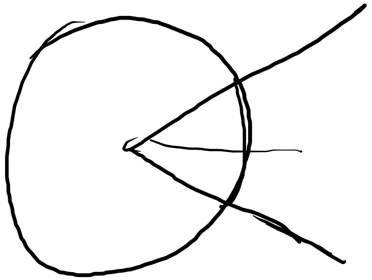
$$= \boxed{abc\,\rho^2\operatorname{sen}\varphi}$$

Vol ellissoide di semiassi a, b, c

$$\iiint_E dm = \int_0^1\left(\int_0^{2\pi}\left(\int_0^{\pi} abc\,\rho^2\operatorname{sen}\varphi\, d\varphi\right)d\vartheta\right)d\rho =$$

$$= abc\cdot\frac{1}{3}\cdot 2\pi\cdot \underbrace{\Big[-\cos\varphi\Big]_0^{\pi}}_{=2} = \frac{4}{3}\pi\, abc$$





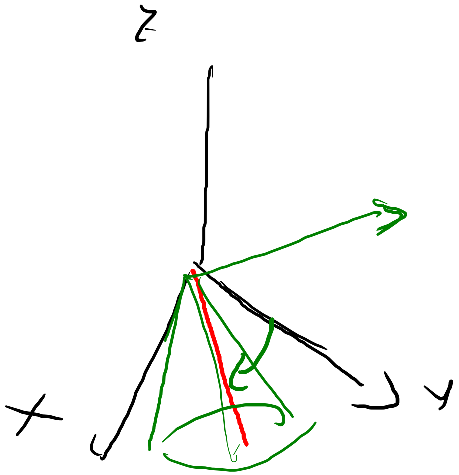
$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\vartheta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \\ z = \rho \cos\varphi \end{cases}$$

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\vartheta \\ y = \rho \cos\varphi \\ z = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \end{cases}$$

$x = x_0 + \rho \operatorname{sen}\varphi \cos\vartheta$

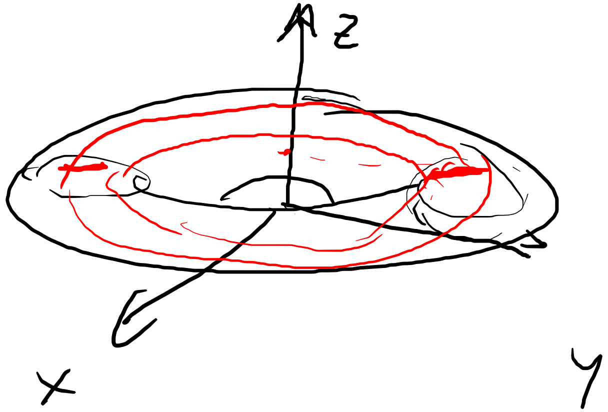
$y = y_0 + \ldots$

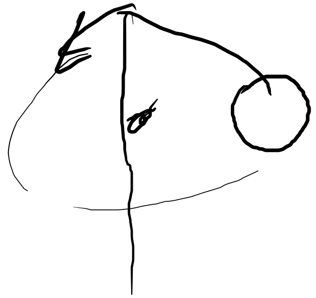
$z = z_0 + \ldots$

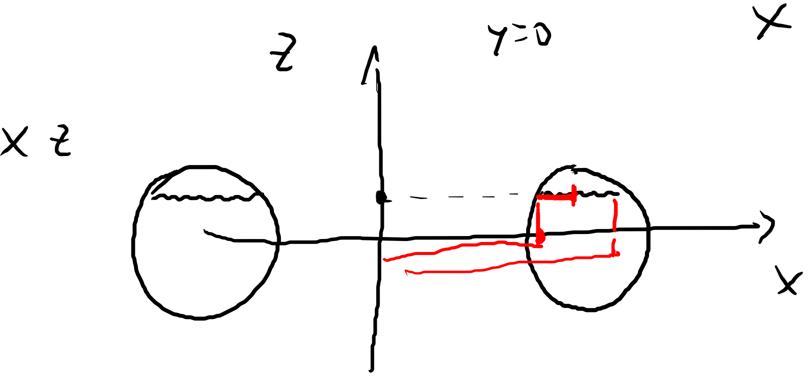


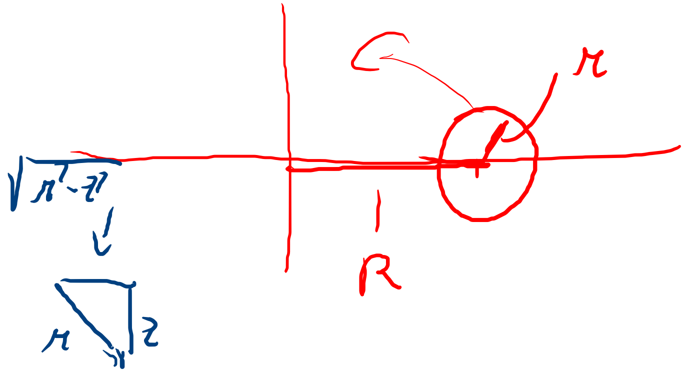
Volume del toro

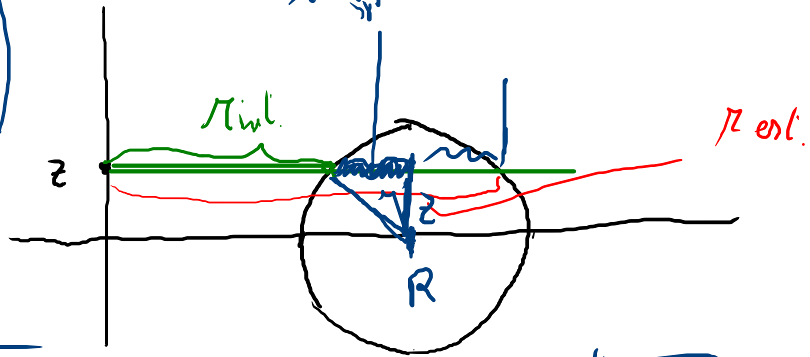
Per sezioni rispetto all'asse z









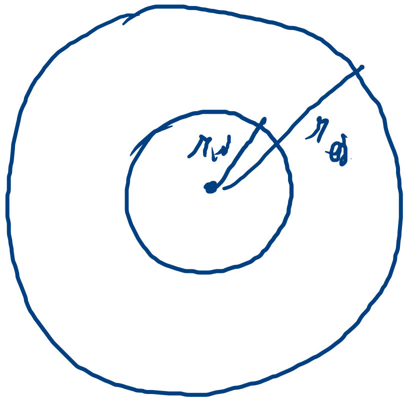


$S_z = ?$ corona circolare con raggio interno

$r\ int = \underline{R - \sqrt{r^2 - z^2}}$

$r\ est. = \underline{R + \sqrt{r^2 - z^2}}$

(raggio interno $R-\sqrt{r^2-z^2}$ e raggio esterno $R+\sqrt{r^2-z^2}$



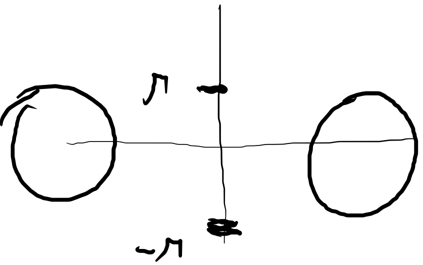
$$\text{Area} = \pi\left(r_{est}^2 - r_{int}^2\right)$$

$$\text{Area } S_z = \pi\left(\left[R+\sqrt{r^2-z^2}\right]^2 - \left[R-\sqrt{r^2-z^2}\right]^2\right)$$

$$= \pi\left(\cancel{R^2} + \cancel{(r^2-z^2)} + 2R\sqrt{r^2-z^2} - \cancel{R^2} - \cancel{(r^2-z^2)} + 2R\sqrt{r^2-z^2}\right)$$

$$= 4\pi R\sqrt{r^2-z^2}$$

$$\text{Volume} = \iiint_E dm = \int_{-r}^{r}\left(\iint_{S_z} dx\,dy\right)dz = \int_{-r}^{r} 4\pi R\sqrt{r^2-z^2}\,dz =$$



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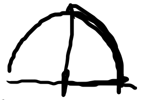
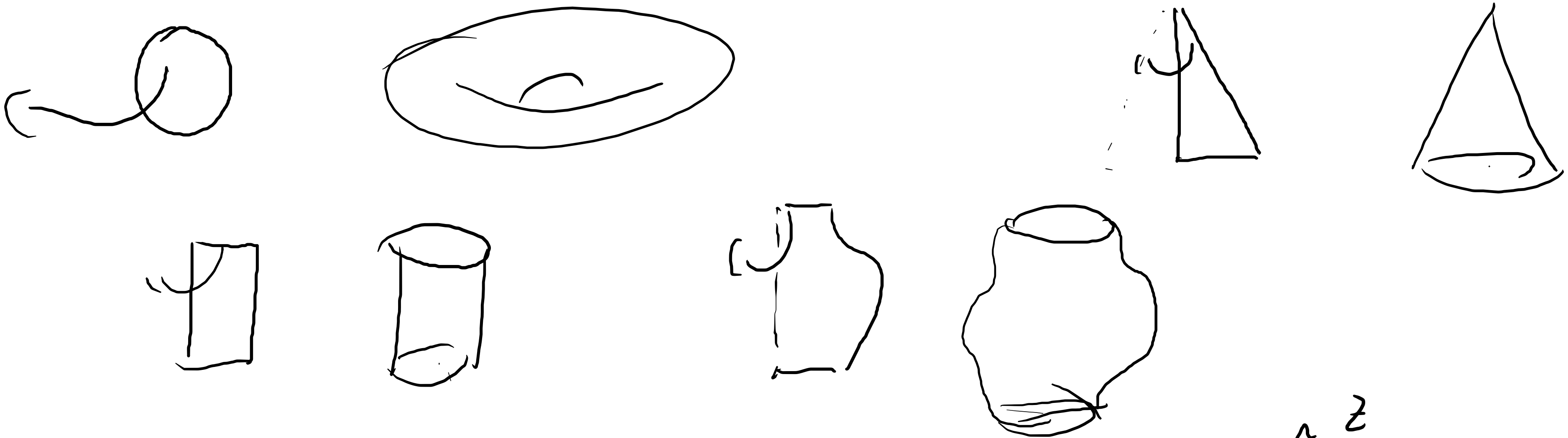
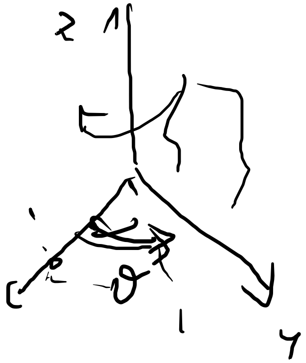


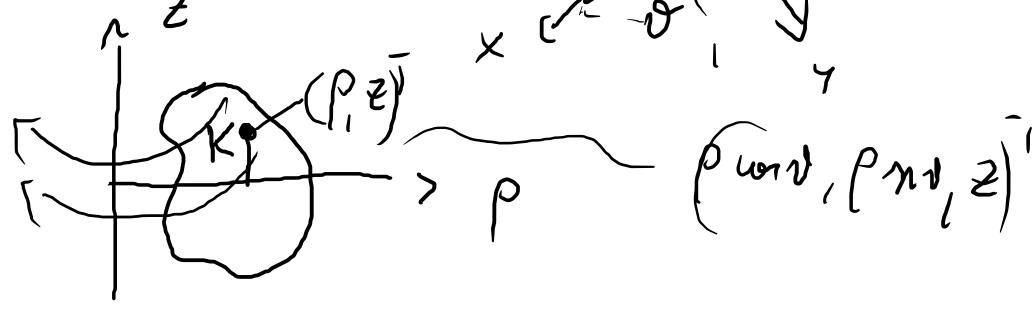
$$= 8\pi R r\int_0^r \sqrt{1-\left(\tfrac{z}{r}\right)^2}\,dz = 8\pi r^2 R\int_0^1 \sqrt{1-u^2}\,du = \boxed{2\pi^2 r^2 R}$$

$u = \frac{z}{r}$ $\quad du = \frac{1}{r}dz$ $\quad = \frac{1}{4}\pi$

Il toro è un esempio di solido di rotazione







Sia K una figura del piano ρz con $\rho \geq 0$

Il solido ottenuto ruotando K intorno all'asse z

si può rappresentare come segue:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \\ z = z \end{cases}$$

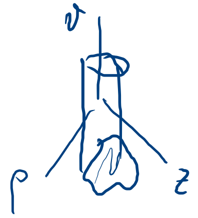
$$(\rho, z)^T \in K \qquad \theta \in [0, 2\pi]$$

Calcoliamo il volume del solido

$$S = \{ (x,y,z)^T \in \mathbb{R}^3 : \underbrace{x = \rho\cos\vartheta,\ y = \rho \operatorname{sen}\vartheta,\ (\rho,z)^T \in K}\}$$

Cambio di variabili $\Phi(\rho,\vartheta,z) = (\underset{x}{\rho\cos\vartheta}, \underset{y}{\rho\operatorname{sen}\vartheta}, \underset{z}{z})^T$

$$J\Phi = \begin{pmatrix} \cos\vartheta & -\rho\operatorname{sen}\vartheta & 0 \\ \operatorname{sen}\vartheta & \rho\cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boxed{\det J\Phi = \rho}$$



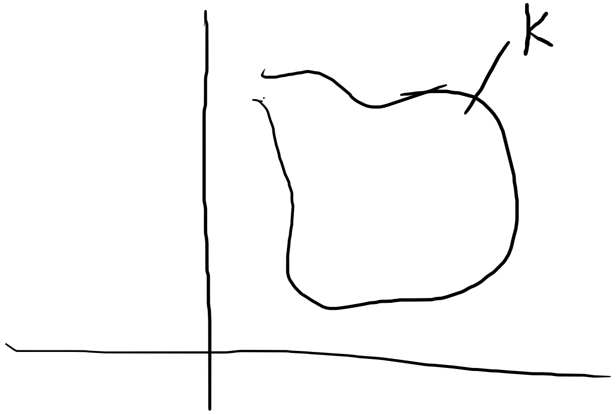
$$\text{Volume di } S = \iiint_S dx\,dy\,dz = \boxed{\iiint_{\{(\rho,\vartheta,z)^T : (\rho,z)^T \in K,\ \vartheta\in[0,2\pi]\}} \rho\, d\rho\, d\vartheta\, dz = *}$$

dominio normale rispetto al ~~piano~~ ρz

$0 \le \vartheta \le 2\pi$

$$* = \iint_K \left(\int_0^{2\pi} \rho\, d\vartheta \right) d\rho\, dz = 2\pi \iint_K \rho\, d\rho\, dz$$

$$\boxed{2\pi \iint_K x\, dx\, dz} = \underbrace{2\pi \cdot \text{Area}(K) \cdot \hat{x}}$$



$$\frac{1}{M}\iint_E x\,dm = \hat{x}$$

$$M = \iint$$

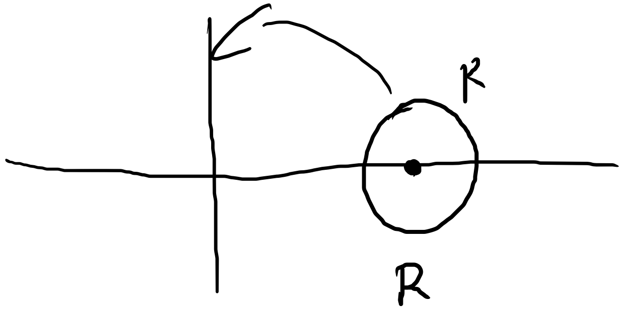
$$\frac{1}{M}\iint$$

Teorema di Pappo-Guldino per i volumi

Sia K un insieme misurabile nel semipiano xz con $x \geq 0$, S il solido di rotazione ottenuto ruotando K attorno all'asse z di un angolo $\alpha \in \,]0, 2\pi]$.
Sia A l'area di K, $\hat{x}$ l'ascissa del baricentro di K. Allora il volume di S è

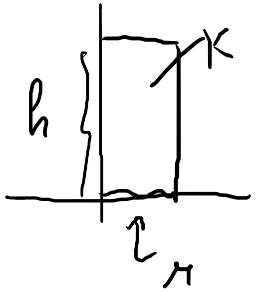
$$\boxed{\mathrm{Vol}(S) = \alpha \cdot A \cdot \hat{x}}$$

Toro



$$\text{Volume} = 2\pi \cdot \pi r^2 R = 2\pi^2 r^2 R$$

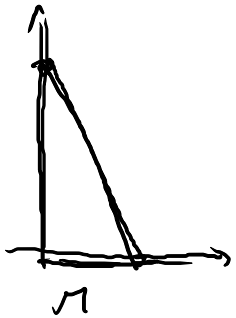
cilindro



Area $r \cdot h$ $\quad \hat{x} = \frac{1}{2} r$

$$\text{Vol} = 2\pi \cdot \frac{1}{2} r \cdot r h = \pi r^2 h$$

cono



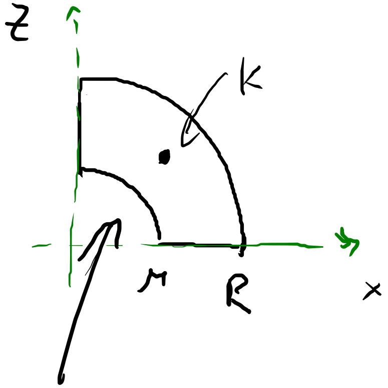
$$\text{Vol} = 2\pi \cdot \frac{1}{2} r h \cdot \hat{x} = \pi r^2 h \frac{1}{3}$$

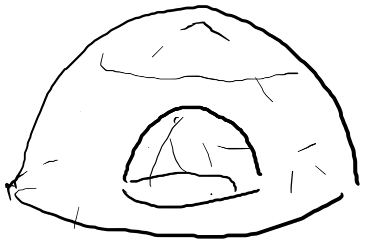
$$\hat{x} \overset{?}{=} \frac{1}{3} r$$

$$\text{Vol. cono} \quad \frac{1}{3}\pi r^2 h = 2\pi \cdot \frac{1}{2} r h \cdot \hat{x}$$

$$\hat{x} = \frac{1}{3} r$$

Baricentro di una lamina omogenea a forma di quarto di corona circolare





$$\mathrm{Vol} = 2\pi \cdot \mathrm{Area} \cdot \hat{x}$$

$$\frac{2}{3}\cancel{\pi}(R^3-r^3) = \cancel{2\pi}\cdot \frac{1}{\cancel{4}_{2}}\pi(R^2-r^2)\cdot \hat{x} \qquad \hat{x} = \frac{4}{3\pi}\,\frac{R^3-r^3}{R^2-r^2}$$

$$\hat{y} = \hat{x}$$

$$\hat{x} = \frac{1}{m}\iint_K x\,dx\,dy = \frac{4}{\pi(R^2-r^2)}\cdot\int_0^{\pi/2}\left(\int_r^R \rho\cos\theta\cdot\rho\,d\rho\right)d\theta =$$

$$= \frac{4}{\pi(R^2-r^2)}\cdot 1\cdot \frac{1}{3}(R^3-r^3)$$

$$K = \{(\rho,\theta) : r \le \rho \le R,\ \theta\in[0,\tfrac{\pi}{2}]\}$$

Integrali generalizzati

f limitata, E limitato

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = ?$$

$$\iint_{B(0,1)} \frac{1}{x^2+y^2}\,dx\,dy = ?$$

$N = 1$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx \quad ?$$

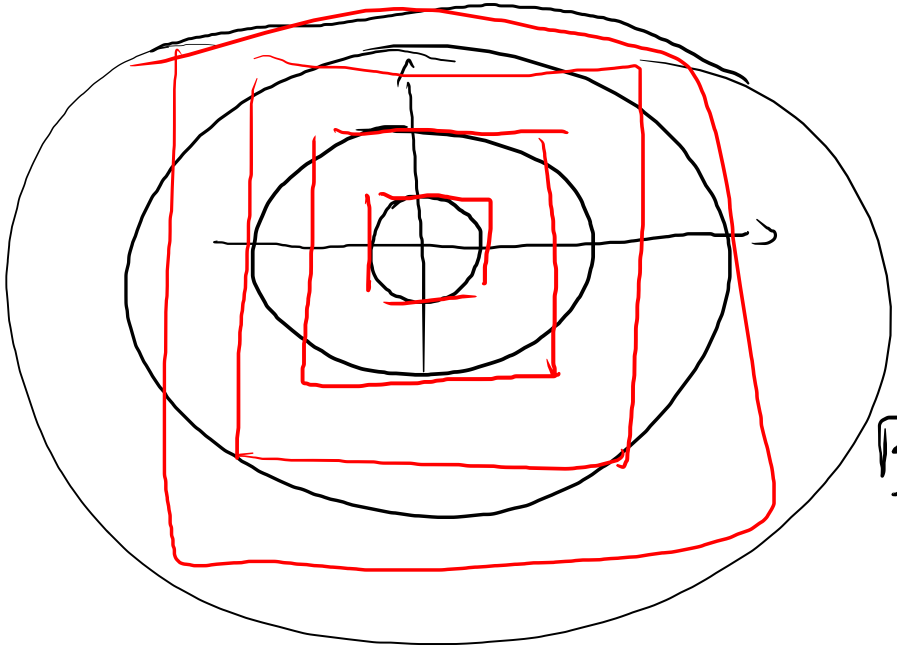
$$\int_{-\infty}^{+\infty} \operatorname{sen} x\,dx = ?$$

$$\int_{-\infty}^{0} \operatorname{sen} x\,dx + \int_0^{+\infty} \operatorname{sen} x\,dx$$

non è integrabile

$$\lim_{n\to+\infty} \int_{-n}^{n} \operatorname{sen} x\,dx = 0$$

$$\iint_{\mathbb{R}^2}$$



$B(0,n)$

$$\lim_{n \to +\infty} \iint_{B(0,n)} f(x)\,dx$$

$$[-n,n] \times [-n,n]$$

Definizione Sia $E \subseteq \mathbb{R}^n$, $f: E \to \mathbb{R}$

Una successione di insiemi misurabili $(A_n)_n$ si dice una successione invadente di E adatta a f se

1) $A_n \subseteq A_{n+1} \subseteq E \quad \forall n$

2) $f|_{A_n}$ è integrabile

3) Per ogni misurabile $M \subseteq E$ si ha che

$$\lim_{n \to +\infty} m(M \setminus A_n) = 0$$