23 Nov

<u>Lemma</u> $\Gamma(t,x)$ C^∞ in $\mathbb{R}_t \times \mathbb{R}^d$

con $|\Gamma(t,x)| \leq C(\sqrt{t} + |x|)^{-d-1}$

~~A~~llora se pongo

$$B(u,v) = \int_0^t \Gamma(t-t') * (u(t') v(t'))\, dt'$$

allora $|B(u,v)(t)|_{L^r_x} \leq C\, t^{-\frac{d}{2}\left(\frac{1}{d} - \frac{1}{r}\right)} |u|_{K_p(t)} |v|_{K_q(t)}$ $(*)$

se

$$0 < \frac{1}{p} + \frac{1}{q} \leq 1$$

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} < \frac{1}{d} + \frac{1}{r}$$

$$|B(u,v)|_{L^r_x} \leq \int_0^t |\Gamma(t-t') * (u(t')v(t'))|_{L^r_x}\, dt'$$

per $\frac{1}{r}+1=\frac{1}{a}+\frac{1}{\beta}$ $\qquad \frac{1}{\beta}=\frac{1}{p}+\frac{1}{q}\leq 1$ ipotesi

$$\leq \int_0^t |\Gamma(t-t')|_{L^a_x}\, |u(t')v(t')|_{L^\beta_x}\, dt'$$

$$\lesssim \int_0^t |\Gamma(t-t')|_{L^a_x}\, |u(t')|_{L^p_x}\, |v(t')|_{L^q_x}\, dt'$$

$$\frac{1}{a}=1+\underbrace{\frac{1}{r}-\frac{1}{p}-\frac{1}{q}}_{\leq 0}$$

$\boxed{1\geq\frac{1}{a}\geq 0}$ $\qquad \frac{1}{r}-\frac{1}{p}-\frac{1}{q}\geq -1$

$\frac{1}{r}\leq\frac{1}{p}+\frac{1}{q}$

$\frac{1}{r}\geq\frac{1}{p}+\frac{1}{q}-1$

$\frac{1}{r}\geq 0 = 1-1 \geq \left(\frac{1}{p}+\frac{1}{q}\right)-1$

$$|\Gamma(t-t')|_{L^a_x} \lesssim \left|\left((t-t')^{\frac{1}{2}}+|x|\right)^{-d-1}\right|_{L^a_x} = (t-t')^{-\frac{d+1}{2}}\left|\left(1+\frac{|x|}{(t-t')^{\frac{1}{2}}}\right)^{-d-1}\right|_{L^a_x}$$

$$=(t-t')^{-\frac{d+1}{2}}\left((t-t')^{\frac{1}{2}}\right)^{\frac{d}{a}}\left|(1+|x|)^{-d-1}\right|_{L^a_x}$$

$$=(t-t')^{-\frac{d}{2}-\frac{1}{2}+\frac{d}{2}\frac{1}{a}}=(t-t')^{\cancel{-\frac{d}{2}}-\frac{1}{2}+\frac{d}{2}\left(\cancel{1}+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}$$

$$|\Gamma(t-t')|_{L^a_x}\lesssim (t-t')^{-\frac{1}{2}+\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}$$

$$|B(u,v)|_{L^r} \le \int_0^t |\Gamma(t-t')|_{L^d_x}\, |u(t')|_{L^p_x}\, |v(t')|_{L^q_x}\, dt'$$

$$\lesssim \int_0^t (t-t')^{\boxed{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)}}\; t'^{-\frac{d}{2}\left(\frac{1}{d}-\frac{1}{p}\right)}\; t'^{-\frac{d}{2}\left(\frac{1}{d}-\frac{1}{q}\right)}\, dt$$

$$\underbrace{\sup_{[0,t]} t'^{\frac{d}{2}\left(\frac{1}{d}-\frac{1}{p}\right)} |u(t')|_{L^p_x}}_{|u|_{K_p}(t)} \qquad \underbrace{\sup_{[0,t]} t'^{\frac{d}{2}\left(\frac{1}{d}-\frac{1}{q}\right)} |v(t')|_{L^p_x}}_{|v|_{K_q}(t)}$$

$$= C\; t^{-\frac{d}{2}\left(\frac{1}{d}-\frac{1}{r}\right)}$$

$$\int_0^t (t-t')^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)}\; t'^{-\frac{d}{2}\left(\frac{2}{d}-\frac{1}{p}-\frac{1}{q}\right)}\, dt$$

$$-\frac{d}{2}\left(\frac{2}{d}-\frac{1}{p}-\frac{1}{q}\right) > -1 \qquad \cancel{\frac{2}{d}} - \frac{1}{p} - \frac{1}{q} < \cancel{\frac{2}{d}}$$

$$\cancel{-\frac{1}{2}} - \frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right) > \cancel{-1} - \frac{1}{2} \qquad \frac{1}{p}+\frac{1}{q} > 0$$

$$\frac{1}{p}+\frac{1}{q}-\frac{1}{r} < \frac{1}{d}$$

$$\frac{1}{p}+\frac{1}{q} < \frac{1}{r}+\frac{1}{d}$$

$$\int_0^t (t-t')^{-\alpha}\,(t')^{-\beta}\,dt' \qquad \alpha,\beta<1$$

$$= C(\alpha,\beta)\; t^{1-\alpha-\beta}$$

$$t' = t s \qquad dt' = t\,ds$$

$$\int_0^1 (t-ts)^{-\alpha}\,(ts)^{-\beta}\,t\,ds =$$

$$= \int_0^1 (1-s)^{-\alpha}\, s^{-\beta}\,ds \;\; t^{1-\alpha-\beta}$$

$$\int_0^t (t-t')^{-\frac12-\frac d2\left(\frac1p+\frac1q-\frac1r\right)} \quad t^{1-\frac d2\left(\frac2d-\frac1p-\frac1q\right)}\,dt$$

$$= t^{\cancel{1}-\left(\frac12+\frac d2\left(\cancel{\frac1p}+\cancel{\frac1q}-\frac1r\right)+\frac d2\left(\cancel{\frac2d}-\cancel{\frac1p}-\cancel{\frac1q}\right)\right)}$$

vogliamo $t^{-\frac d2\left(\frac1d-\frac1r\right)}$

$$= t^{-\frac12+\frac d2\frac1r} = t^{-\frac d2\frac1d+\frac d2\frac1r} = t^{-\frac d2\left(\frac1d-\frac1r\right)}$$

<u>Teor</u> Le soluzioni $u\in C^0([0,T),L^3)$ appartengono a $C^\infty((0,T)\times\mathbb{R}^3,\mathbb{R}^3)$

$$u\in L^\infty([0,T),L^3)$$

Lemma Sia $f \in \mathcal{S}(\mathbb{R}^3)$ e sia $u \in S'(\mathbb{R}^3)$ t.c. $-\Delta u = f$. Allora $u = K * f + h$

con $K(x) = \frac{1}{4\pi|x|}$ e $h(x)$ un polinomio armonico

$(\Delta h = 0)$

Dim Sia $h \in S'(\mathbb{R}^3)$ t.c. $-\Delta h = 0 \Leftrightarrow |\xi|^2 \hat{h} = 0$

$\Rightarrow \operatorname{supp} \hat{h} = 0 \Leftrightarrow \hat{h} = \sum_{|\alpha| \leq k} a_\alpha \partial_\xi^\alpha \delta_0$.

$\Rightarrow h(x)$ è un polinomio, in particolare un polinomio armonico.

$$\hat{v} = \frac{1}{|\xi|^2} \hat{f} \qquad -\Delta v = f$$

$$\mathcal{F}^*(|x|^{-\gamma})(\xi) = |\xi|^{\gamma - d} \frac{2^{\frac{d-\gamma}{2}} \Gamma\left(\frac{d-\gamma}{2}\right)}{2^{\frac{\gamma}{2}} \Gamma\left(\frac{\gamma}{2}+1\right)} = |\xi|^{-1} \frac{2^{\frac{1}{2}}\sqrt{\pi}}{2}$$

$d = 3$, $\gamma = 2$

$$= \sqrt{\frac{\pi}{2}} \, |\xi|^{-1}$$

$$\Gamma\left(\frac{\gamma}{2}+1\right) = \Gamma(2) = 1$$

$$\Gamma\left(\frac{d-\gamma}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\frac{1}{|\xi|^2} \longrightarrow \sqrt{\frac{\pi}{2}} \, \frac{1}{|x|}$$

$$v = \left(\frac{1}{|\xi|^2} \hat{f} \right)^{\vee} = \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\pi}{2}} \frac{1}{|x|} * f =$$

$$= \frac{1}{4\pi} \frac{1}{|x|} * f$$

Per linearità $u = \frac{1}{4\pi} \frac{1}{|x|} * f + h$.

Dato $u \in S'(\mathbb{R}^3, \mathbb{R}^3)$, $w \doteq \nabla \times u$

$\operatorname{div} w = 0$

<u>Lemma</u> Sia $u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$ con $p \in (1,3)$ e $\operatorname{div} u = 0$ e sia $w = \nabla \times u$. Allora $u = Tw$ dove

$$Tw = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times w(y)\, dy$$

$$\int_\Omega \nabla \cdot E \, dx = \int_{\partial\Omega} E \cdot N \, dS$$

$$E_j = \frac{1}{|x-y|} w_k(y)$$

<u>Dim</u> Ricordando $\Delta u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u)$

$$\Delta u = -\nabla \times w$$

Incominciamo con $w \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$

$$\frac{1}{|x|} * \partial_j w_k = \left(\partial_j \frac{1}{|x|} \right) * w_k$$

$$\frac{1}{|x|} * \partial_j w_k = \int_{\mathbb{R}^3} \frac{1}{|x-y|} \partial_j w_k(y)\, dy = \lim_{\varepsilon \to 0^+} \int_{|x-y| \geq \varepsilon} \frac{1}{|x-y|} \partial_j w_k(y)\, dy =$$

$$= \lim_{\varepsilon \to 0^+} \left[-\int_{|x-y| \geq \varepsilon} \partial_{y_j} \frac{1}{|x-y|} w_k(y)\, dy + \int_{|x-y| = \varepsilon} \cancel{\frac{x_j - y_j}{|x-y|^2} w_k(y)} \, dS \right]$$

$$\partial_{y_j} \frac{1}{|x-y|} = -\partial_{x_j} \frac{1}{|x-y|}$$

$$\frac{1}{|x|} * \partial_j w_k = \partial_j \frac{1}{|x|} * w_k$$

$\epsilon_{123} = 1$

$\epsilon_{ijk} = \pm 1$ a seconda della ~~rout~~ ruota di (i, j, k) se è una permutazione di $(1,2,3)$

$$\Delta u = -\nabla \times w = -e_i \, \epsilon_{ijk} \, \partial_j w_k$$

$$v = (-\Delta)^{-1} \epsilon_{ijk} \, e_i \, \partial_j w_k = e_i \frac{\epsilon_{ijk}}{4\pi} \frac{1}{|x|} * \partial_j w_k =$$

$$= e_i \frac{\epsilon_{ijk}}{4\pi} \partial_j \frac{1}{|x|} * w_k =$$

$$= -e_i \frac{\epsilon_{ijk}}{4\pi} \frac{x_j}{|x|^3} * w_k =$$

$$= -\frac{1}{4\pi} e_i \epsilon_{ijk} \int_{\mathbb{R}^3} \frac{x_j - y_j}{|x-y|^3} w_k(y) \, dy$$

$$v = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times w(y) \, dy = Tw$$

$$u = Tw + h \qquad h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

h_j sono polinomi armonici

$$\Delta h = 0$$

$0 < \gamma < d$

$$w \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$$

$$L^p(\mathbb{R}^3, \mathbb{R}^3)$$

$$\left| |x|^{-\gamma} * w \right|_{L^q(\mathbb{R}^d)} \le C \, |w|_{L^p(\mathbb{R}^d)}$$

$$|Tw|_{L^q} \le C \, |w|_{L^p}$$

$$\frac{1}{q} = \frac{1}{p} - \frac{d-\gamma}{d}$$

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$$

$u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$

$u \in L^q(\mathbb{R}^3, \mathbb{R}^3)$

$h \in L^q(\mathbb{R}^3, \mathbb{R}^3)$ $\qquad 1 < q < \infty$

$h \equiv 0$

$\boxed{u = Tw}$ $\qquad w \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$

In generale $u \in W^{1,p}(\mathbb{R}^3) \Rightarrow w \in L^p(\mathbb{R}^3)$

$w_n \to w$ $\qquad \boxed{u_n = \boxed{Tw_n}} \to Tw = \tilde{u}$ in $L^q(\mathbb{R}^3)$

$\cap$

$C_{c\sigma}^\infty(\mathbb{R}^3)$

$\boxed{\nabla \times \tilde{u} = w}$

$\Delta(u - \tilde{u}) = -\nabla \times (u - \tilde{u}) = 0$

$u - \tilde{u} = h \in L^q(\mathbb{R}^3) \Rightarrow h \equiv 0$

$w = \nabla \times u$

$\tilde{u} = Tw$ $\qquad \boxed{\nabla \times \tilde{w} = w}$ $\quad L^p$

$\qquad \nabla \times (Tw_n) = w_n$

$w \in C_c^\infty(\mathbb{R}^3)$

$\tilde{u} = Tw = e_i \, \epsilon_{ijk} \frac{1}{4\pi} \frac{1}{|x|} * \partial_j w_k =$

$= \frac{1}{4\pi} \frac{1}{|x|} * (\nabla \times w)$

$\Delta \tilde{u} = \nabla(\nabla \cdot \tilde{u}) - \nabla \times (\nabla \times \tilde{u}) = -\nabla \times (\nabla \times \tilde{u})$

$= -\nabla \times w$ $\qquad w = \nabla \times \tilde{u}$