Notion of mapping

X, Y sets

$f : X \to Y$ is a mapping if $\forall x \in X, \exists ! \, y \in Y$ such that $y = f(x)$

X domain of f, Y codomain

$f(X)$ image set of f $\quad f(X) = \{ y \in Y : \exists \, x \in X$ such that $f(x) = y \}$

If $X, Y \subset \mathbb{R}$, $f : X \to Y$ is called a real-valued function in one real variable.

Notions of injectivity, surjectivity, bijectivity

$f: A \to B \qquad A, B \subset \mathbb{R} \qquad$ (f function)

f is INJECTIVE if, for all $x_1, x_2 \in A$

$$x_1 \neq x_2 \longrightarrow f(x_1) \neq f(x_2)$$

f is SURJECTIVE if $f(A) = B$ (this means that for all $y \in B$, $\exists\, x \in A$ such that $f(x) = y$)

f is BIJECTIVE if f is both injective and surjective.

Property: if $f: A \to B$ is bijective, then f is invertible.

Notion of invertibility

$f : A \to B \qquad A, B \subset \mathbb{R}$

f is invertible if there exists its inverse function

$f^{-1} : B \to A \qquad \forall x \in A$

$$x = f^{-1}(y) \longleftrightarrow y = f(x)$$

Exercise 2.4.1 (from the book) about domains

$h(x) = \sqrt{x+1} + \sqrt{3-2x}$

$\left(\sqrt{\cdot} : [0, +\infty[\longrightarrow [0, +\infty[\right)$

$\mathrm{dom}(h) = \{ x \in \mathbb{R} : x+1 \geq 0 \text{ and } 3-2x \geq 0 \} = \left[-1, \frac{3}{2} \right]$

$\begin{cases} x+1 \geq 0 \\ 3-2x \geq 0 \end{cases}$ $\quad \begin{cases} x \geq -1 \\ x \leq \frac{3}{2} \end{cases}$ $\quad -1 \leq x \leq \frac{3}{2}$

$j(x) = \dfrac{1+\sqrt{x}}{x^3\left(1-\sqrt{3x}\right)}$

$\mathrm{dom}(j) = \{ x \in \mathbb{R} : x \geq 0 \text{ and } 3x \geq 0 \text{ and } x^3\left(1-\sqrt{3x}\right) \neq 0 \}$

$$\begin{cases} x \geq 0 \\ \cancel{3x \geq 0} \\ x^3\left(1-\sqrt{3x}\right) \neq 0 \end{cases}$$

$$x^3\left(1-\sqrt{3x}\right) \neq 0 \iff x^3 \neq 0 \text{ and } 1-\sqrt{3x} \neq 0$$

(by the null factor law)

$$\begin{cases} x \geq 0 \\ x^3 \neq 0 \\ 1-\sqrt{3x} \neq 0 \end{cases}$$

$$x^3 = 0 \iff x = 0$$

$$x^3 \neq 0 \iff x \neq 0$$

$$\begin{cases} x \geq 0 \\ x \neq 0 \\ \sqrt{3x} \neq 1 \end{cases}$$

$$\sqrt{3x} = 1 \iff \left(\sqrt{3x}\right)^2 = 1^2$$

$$\iff 3x = 1 \iff x = \frac{1}{3}$$

$$\begin{cases} x \geq 0 \\ x \neq 0 \\ x \neq \frac{1}{3} \end{cases}$$

$$0 < x < \frac{1}{3} \quad \vee \quad x > \frac{1}{3}$$

$$\boxed{\text{dom}(j) = \left]0, \frac{1}{3}\right[\cup \left]\frac{1}{3}, +\infty\right[}$$

$$k(x) = \frac{x^6 - \sqrt{\pi}}{\sqrt{x+7}}$$

$$\boxed{\text{dom}(k) =]-7, +\infty[}$$

$$\begin{cases} x + 7 \geq 0 \\ \sqrt{x+7} \neq 0 \end{cases} \qquad \begin{cases} x \geq -7 \\ x \neq -7 \end{cases} \qquad x > -7$$

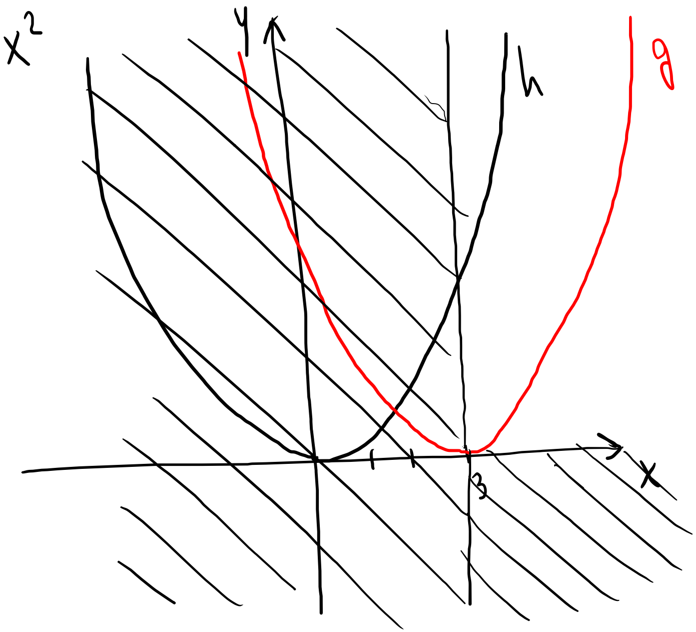
$$\sqrt{x+7} = 0 \Leftrightarrow x + 7 = 0 \Leftrightarrow x = -7$$

Exercise 2.4.6 (from the book) about bijective functions

$g : \mathbb{R} \to \mathbb{R}$

$g(x) = x^2 - 6x + 9 = (x-3)^2$

$h(x) = x^2$



g is not injective
(h is not injective)

g is not surjective
(g does not attain any negative value)

$\longrightarrow$ g is not bijective

$h(x) = x^2 - 6x + 9$

$h : [3, +\infty[\longrightarrow [0, +\infty[$

(with respect to g, the law does not change, what change are the domain which is $[3, +\infty[$ and the codomain which is $[0, +\infty[$)

$h : [3, +\infty[\longrightarrow [0, +\infty[$

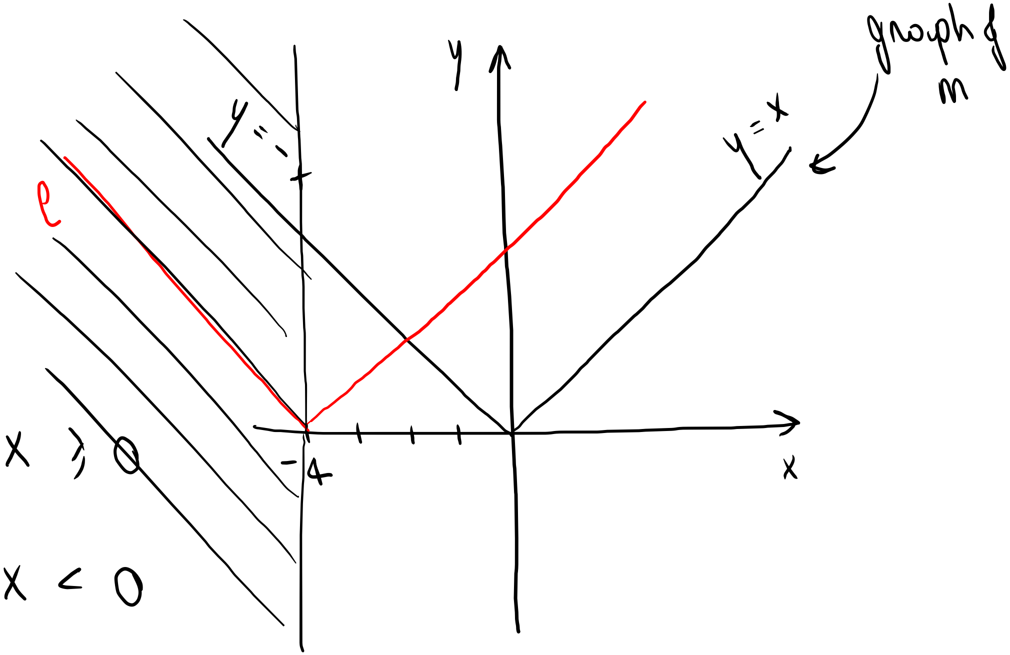
$h(x) = (x-3)^2$ is both injective and surjective then it is bijective

then it is invertible.

$\ell(x) = \sqrt{(x+4)^2} = |x+4|$

$\ell : \mathbb{R} \longrightarrow [0, +\infty[$

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$m(x) = |x|$

f is surjective (f takes all the non-negative values)

f is not injective

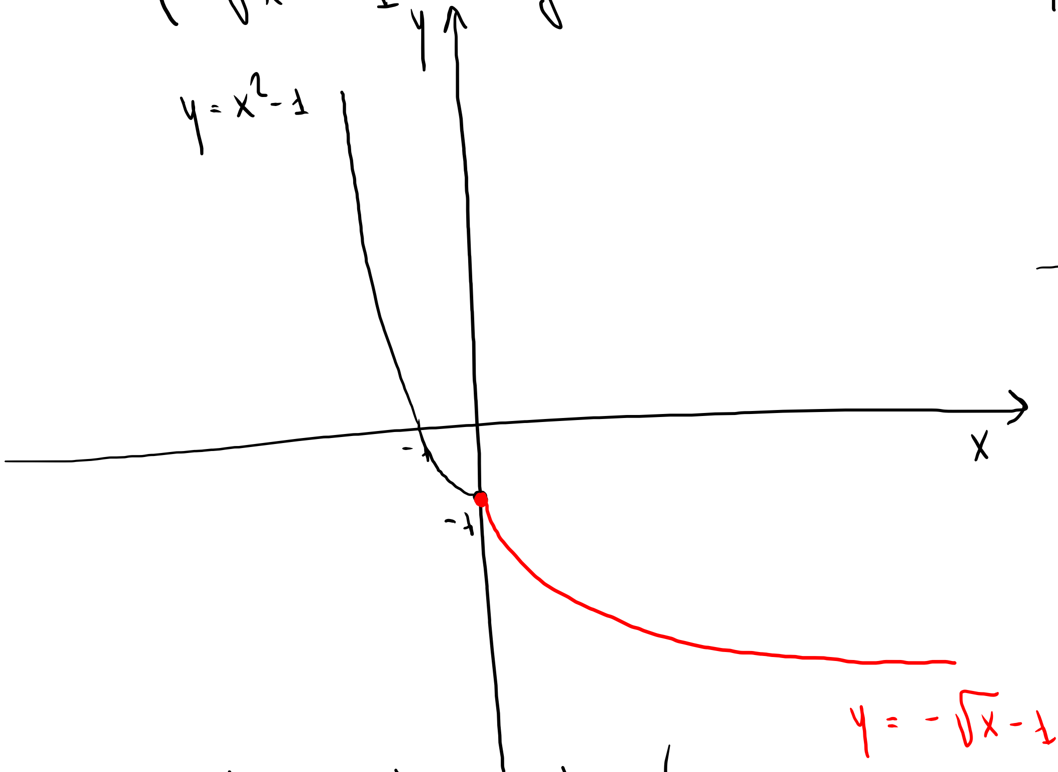
$\longrightarrow f$ is not bijective

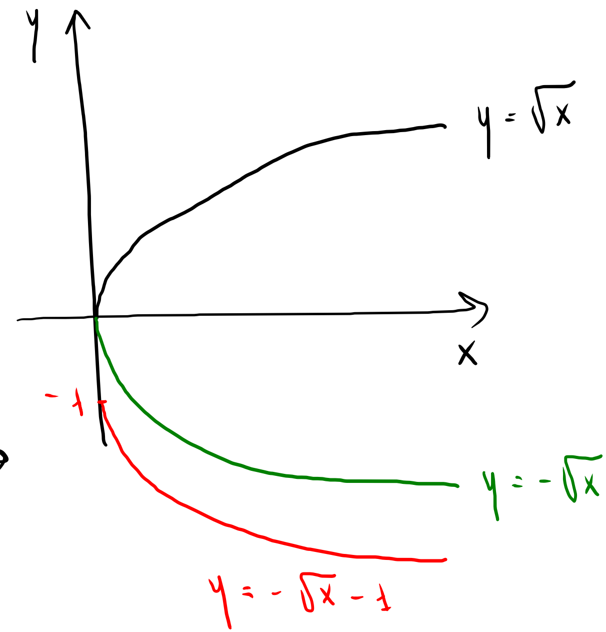
$f|_{[-4,+\infty[} : [-4,+\infty[\longrightarrow [0,+\infty[$ is both injective and surjective, so it is bijective.

Exercise 1 (of the paper on Moodle, November 16, 2020)

$$g(x) = \begin{cases} x^2 - 1 & \text{if} \quad x < 0 \\ -\sqrt{x} - 1 & \text{if} \quad x \geq 0 \end{cases}$$

piecewise function





g is injective and surjective (it is bijective, then it is invertible)

Determination of the inverse function

$x < 0$ $\quad \left(g(x) = x^2 - 1 \longrightarrow y = g(x) > -1 \right)$

$$y = x^2 - 1$$

$$x^2 = y + 1$$

$$-x = |x| = \sqrt{x^2} = \sqrt{y+1}$$

$x < 0$

$x = -\sqrt{y+1} = g^{-1}(y)$ when $y > -1$

$x \geq 0$ $\quad \left(g(x) = -\sqrt{x} - 1 \longrightarrow y = g(x) \leq -1 \right)$

$$y = -\sqrt{x} - 1$$

$$\sqrt{x} = -y - 1$$

$$\left(\sqrt{x}\right)^2 = \left(-y-1\right)^2$$

$x = (y+1)^2 = g^{-1}(y)$ when $y \leq -1$

$g^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$

$$g^{-1}(y) = \begin{cases} (y+1)^2 & \text{if } y \leq -1 \\ -\sqrt{y+1} & \text{if } y > -1 \end{cases}$$