25 Novembre

L. $u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$ $1<p<3$, $\operatorname{div} u = 0$

Allora se $\omega = \nabla \times u$ allora $u = T\omega$

<u>Dim</u> 1) Caso $\omega \in C_c^\infty$

2) Caso generale, $\omega \in L^p(\mathbb{R}^3, \mathbb{R}^3)$

$$\underset{C_c^\infty}{\underset{\Uparrow}{\tilde{\omega}_n}} \to \omega \quad \text{in } L^p$$

$$u_n = T\tilde{\omega}_n \implies u_n \to \tilde{u} = T\omega \qquad L^q(\mathbb{R}^3, \mathbb{R}^3)$$

Vogliamo dimostrare $\tilde{u} = u$

$$\nabla \cdot u_n = 0 = \nabla \cdot T\tilde{\omega}_n = \nabla \cdot (-\Delta)^{-1} \nabla \times \tilde{\omega} =$$

$$= (-\Delta)^{-1} \underbrace{\nabla \cdot \nabla \times \tilde{\omega}}_{0} = 0$$

$$\nabla \cdot \tilde{u} = 0$$

$\mathbb{P}$ proiettore di Leray è un operatore di Calderon Zygmund ed è quindi continuo $L^p \circlearrowleft$ $\forall\ p \in (1,\infty)$

$$\tilde{\omega}_n = \tilde{\omega}_n^{(1)} + \tilde{\omega}_n^{(2)} \qquad \tilde{\omega}_n^{(1)} = \mathbb{P}\,\tilde{\omega}_n$$

$$\tilde{\omega}_n^{(2)} = (1-\mathbb{P})\,\tilde{\omega}_n$$

$$\tilde{\omega}_n \to \omega \implies \tilde{\omega}_n^{(1)} \to \omega \qquad \tilde{\omega}_n^{(2)} \to 0 \quad \text{in } L^p$$

Voglio $\quad \tilde{\omega}_n^{(1)} = \nabla \times u_n \doteq \omega_n \qquad u_n = T\tilde{\omega}_n$

$$-\Delta u_n = \nabla \times \tilde{\omega}_n = \nabla \times \tilde{\omega}_n^{(1)} + \underbrace{\nabla \times \tilde{\omega}_n^{(2)}}_{0}$$

$$-\Delta u_n = -\nabla(\underbrace{\nabla \cdot u_n}_{0}) + \nabla \times \omega_n$$

$$-\Delta u_n = \nabla \times \underline{\omega_n} = \nabla \times \underline{\tilde{\omega}_n^{(1)}}$$

$$\mathbb{P}\,\sqcup = -\Delta^{-1}\,\nabla \times (\nabla \times \sqcup)$$

$$\omega_n = \mathbb{P}\,\omega_n = \mathbb{P}\,\tilde{\omega}_n^{(1)} = \tilde{\omega}_n^{(1)} = \nabla \times u_n \quad \blacksquare$$

$$\nabla \times u_n = \tilde{\omega}_n^{(1)} \xrightarrow{L^p} \omega$$

$\mathcal{D}'$: $\nabla \times u_n \to \nabla \times \tilde{u}$ $\implies$ $\underline{\nabla \times \tilde{u} = \omega = \nabla \times u}$ $\quad u, \tilde{u} \in L^q(\mathbb{R}^3, \mathbb{R}^3)$

$$-\Delta(u-\tilde{u}) = \nabla \times (\underbrace{\nabla \times (u-\tilde{u})}_{0}) = 0$$

$$-\Delta(u-\tilde{u}) = 0 \implies u - \tilde{u} = h \in L^q \qquad h \equiv 0$$

Equazione della vorticità:

$$|u|^2 = u_j u_j$$

$$u_t - \Delta u + u\cdot\nabla u = -\nabla p$$

$$u_n \to u \qquad L^2 \qquad H^1(\mathbb{R}^3)$$

$$(u\cdot\nabla)u = 2^{-1}\nabla|u|^2 - u\times w$$

$$L = (u\cdot\nabla)u = e_i u_j \partial_j u_i$$

$$R = e_i u_j \partial_i u_j - e_i \epsilon_{ijk} u_j (\nabla\times u)_k =$$

$$= e_i u_j \partial_i u_j - e_i \underbrace{\epsilon_{ijk}\epsilon_{i'j'k}}_{(\delta_{ii'}\delta_{jj'}-\delta_{ij'}\delta_{ji'})} u_j \partial_{i'} u_{j'}$$

$$= e_i u_j \partial_i u_j - e_i u_j \partial_i u_j + e_i u_j \partial_j u_i$$

$$\nabla\times\left((u\cdot\nabla)u\right) = \nabla\times\left(\tfrac{1}{2}\nabla|u|^2 - u\times w\right) = -\nabla\times(u\times w)$$

$$= (u\cdot\nabla)w - (w\cdot\nabla)u \qquad H^2 \qquad w = \nabla\times u$$

$$\nabla\times \qquad \partial_t u - \Delta u + (u\cdot\nabla)u = -\nabla p \qquad w = \nabla\times u$$

$$\boxed{\partial_t w - \Delta w + (u\cdot\nabla)w = (w\cdot\nabla)u} \Leftarrow \text{equazione vorticità}$$

$$\langle\ ,\phi\rangle_{L^2_{tx}} \qquad \forall\ \phi\in C_c^\infty((0,\infty)\times\mathbb{R}^3,\mathbb{R}^3)$$

$$\int_0^\infty\left(\langle w,\partial_t\phi\rangle_{L^2_x} + \langle w,\Delta\phi\rangle_{L^2_x} + \langle w, u\cdot\nabla\phi\rangle_{L^2_x} - \langle u, w\cdot\nabla\phi\rangle_{L^2_x}\right)dt' = 0$$

$$\langle u_j\partial_j w,\phi\rangle_{L^2_x} = -\langle w, \partial_j(u_j\phi)\rangle_{L^2_x} = -\langle w,(u\cdot\nabla)\phi\rangle$$

Lemma Sia u una soluzione debole $u \in L^\infty(\mathbb{R}_+, L^2)$, $\nabla u \in L^2(\mathbb{R}_+, L^2)$ e sia $\omega = \nabla \times u$. Allora (u, ω) soddisfa

$$\forall \phi \in C_c^\infty((0,\infty) \times \mathbb{R}^3, \mathbb{R}^3)$$

$$\int_0^\infty \left(\langle \omega, \partial_t \phi \rangle_{L^2_x} + \langle \omega, \Delta \phi \rangle_{L^2_x} + \langle \omega, u \cdot \nabla \phi \rangle_{L^2_x} - \langle u, \omega \cdot \nabla \phi \rangle_{L^2_x} \right) dt' = 0$$

Dim Sia $\phi \in C_c^\infty((0,\infty) \times \mathbb{R}^3, \mathbb{R}^3)$

$$\Rightarrow \quad \nabla \times \phi \in C_{c\sigma}^\infty((0,\infty) \times \mathbb{R}^3, \mathbb{R}^3)$$

$$\Rightarrow \int_{\mathbb{R}_+} \left(-\langle u, (\nabla \times \partial_t \phi) \rangle - \langle u, \Delta \nabla \times \phi \rangle + \langle (u \cdot \nabla) u, \nabla \times \phi \rangle \right) dt' = 0$$

$$= \int_{\mathbb{R}_+} \left(\langle \omega, \partial_t \phi \rangle + \langle \omega, \Delta \phi \rangle + \dots \right)$$

$$\langle (u \cdot \nabla) u, \nabla \times \phi \rangle = -\langle \nabla \times [(u \cdot \nabla) u], \phi \rangle =$$

$$= \langle (\omega \cdot \nabla) u - (u \cdot \nabla) \omega, \phi \rangle$$

$u(t) \in H^2$ q.o.

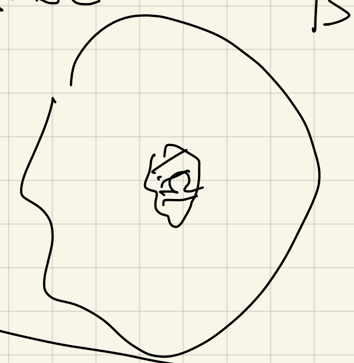
$u(t) \in H^1$ nei quasi ogni t

$u \in L^\infty_t L^2_x \quad \nabla u \in L^2_t L^2_x$

Lemma Sia B un aperto limitato con ∂B liscio, $u \in L^p(B, \mathbb{R}^3)$, $\nabla u \in L^p(B)$ $r \in [1, +\infty]$, $p \in (1, \infty)$.

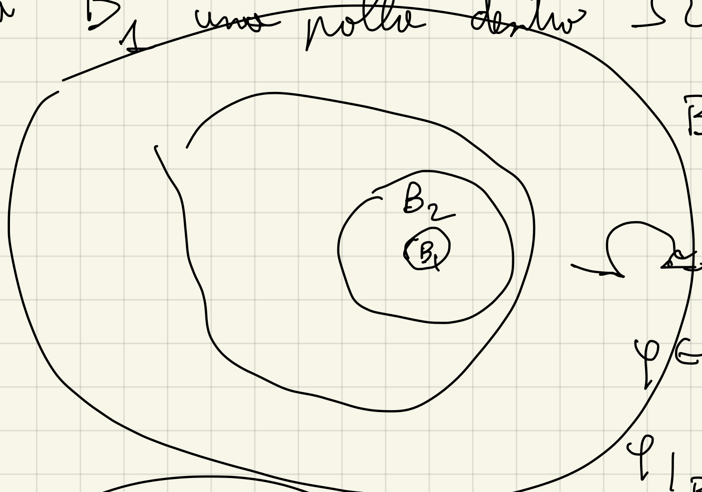
Sia $\Omega \subset \bar{\Omega} \subset\subset B$. Allora

$$u(x) = T(\chi_\Omega w) + h(x) \quad \forall x \in \Omega$$



h è armonica in Ω.

Sia B_1 una palla dentro Ω



$$B_1 \subset \bar{B}_1 \subset\subset B_2 \subset \bar{B}_2 \subset\subset \Omega$$

$$\varphi \in C_c^\infty(\Omega, [0,1])$$

$$\varphi|_{B_2} = \chi_{B_2}$$

$$\tilde{u} = T\, \nabla \times (\varphi u) = T(\varphi w) + T(\nabla\varphi \times u) =$$

$$= T(\chi_\Omega w) + \underbrace{T((\varphi - \chi_\Omega) w) + T(\nabla\varphi \times u)}_{\tilde{h}}$$

$\tilde{h}|_{B_1}$ è armonica $x \in B_1$

$$\Delta_x \tilde{h} = -\frac{1}{4\pi} \int_{\Omega \setminus B_2} \underbrace{\Delta_x \frac{x-y}{|x-y|^3}}_{=0} \times w(y) \left(\varphi(y) - \chi_\Omega(y)\right) dy - \frac{1}{4\pi} \int_{\Omega \setminus B_2} \underbrace{\Delta_x \frac{x-y}{|x-y|^3}}_{=0} \times \left(\nabla\varphi(y) \times u(y)\right) dy$$

$$\Delta_x \frac{x}{|x|^3} = 0 \quad x \neq 0 \qquad \nabla \Delta \frac{1}{r} = 0 \quad \text{per } r \neq 0$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\frac{1}{r} = 0$$

$$\frac{x}{|x|^3} = -\nabla \frac{1}{r}$$

$$\Delta \nabla \frac{1}{r} = 0$$

In B_2 $\quad \nabla \times (u - \tilde{u}) = \nabla \times (\varphi u - T \nabla \times (\varphi u)) = 0$

$$\mathbb{P}(\varphi u) = T \; \nabla \times (\varphi u) = T \; \nabla \times \mathbb{P}(\varphi u) \qquad \cancel{\varphi u} \; \cancel{X}$$

$$= \tilde{u}$$

$$\mathbb{P}(\varphi u) = \tilde{u}$$

$$\nabla \times (\nabla \times \tilde{u})$$

$$\nabla \times (\nabla \times \mathbb{P}(\varphi u))$$

$$-\Delta(\varphi u) = -\nabla(\nabla \cdot (\varphi u)) + \nabla \times (\nabla \times (\varphi u))$$

$$-\Delta \tilde{u} = \nabla \times (\nabla \times \tilde{u})$$

$$-\Delta(\varphi u) = -\nabla(\nabla \cdot (\varphi u)) - \Delta \tilde{u} \qquad \text{in } B_2$$

In B_2 $\quad \Delta(u - \tilde{u}) = 0 \qquad u - \tilde{u} = h_1$ armonici

in B_2 $\quad \tilde{u} = T(\chi_\Omega w) + \tilde{h} \qquad \tilde{h}$ armonico in B_1

$h = u - T(\chi_\Omega w)$ è armonico in B_1

$\mathbb{P}(\varphi u) = T \, \nabla \times \mathbb{P}(\varphi w)$

Prima se $v \in W^{1,p_0}(\mathbb{R}^3, \mathbb{R}^3)$ con $1 < p_0 \le 3$

$\nabla \cdot v = 0$, allora $v = T(\nabla \times v)$

$u \in L^r(B, \mathbb{R}^3)$ $\quad \nabla u \in L^p(B)$

$\varphi u \in L^r(\mathbb{R}^3, \mathbb{R}^3)$ $\quad \nabla(\varphi u) \in L^{\min(r,p)}(\mathbb{R}^3, \mathbb{R}^3)$ $\quad (\alpha)$

$\varphi u \in L^{\min(r,p)}(\mathbb{R}^3, \mathbb{R}^3)$

$\mathbb{P}(\varphi u)$, $\varphi u \in W^{1,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ $\quad 1 < \alpha < 3$ $\quad B$

$\alpha \ge 3$

$\nabla u \in L^p(B)$

$u \in L^r(B) \Rightarrow u \in L^1(B)$

$$\| u - \fint_B u \|_{L^p(B)} \le C_B \| \nabla u \|_{L^p(B)}$$

$\Rightarrow u \in L^p(B)$ $\quad 1 < p$

$\varphi u \in L^p(\mathbb{R}^3, \mathbb{R}^3)$

$\varphi u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$

$1 < p < \infty$

$W^{k,p}(\mathbb{R}^d)$

$1 = k$

$W^{1,p}(\mathbb{R}^d) = \{ f \in \mathcal{S}' : (1-\Delta)^{\frac{1}{2}} f \in L^p \}$

$(1 + |\xi|^2)^{\frac{1}{2}} \hat{f}$