Image Processing for Physicists

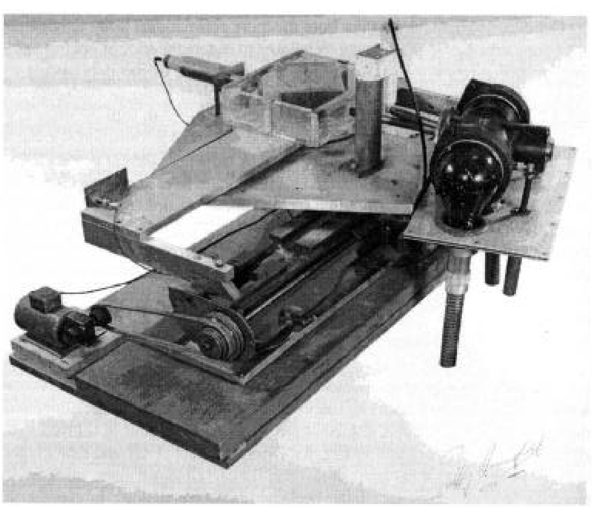
Prof. Pierre Thibault
pthibault@units.it

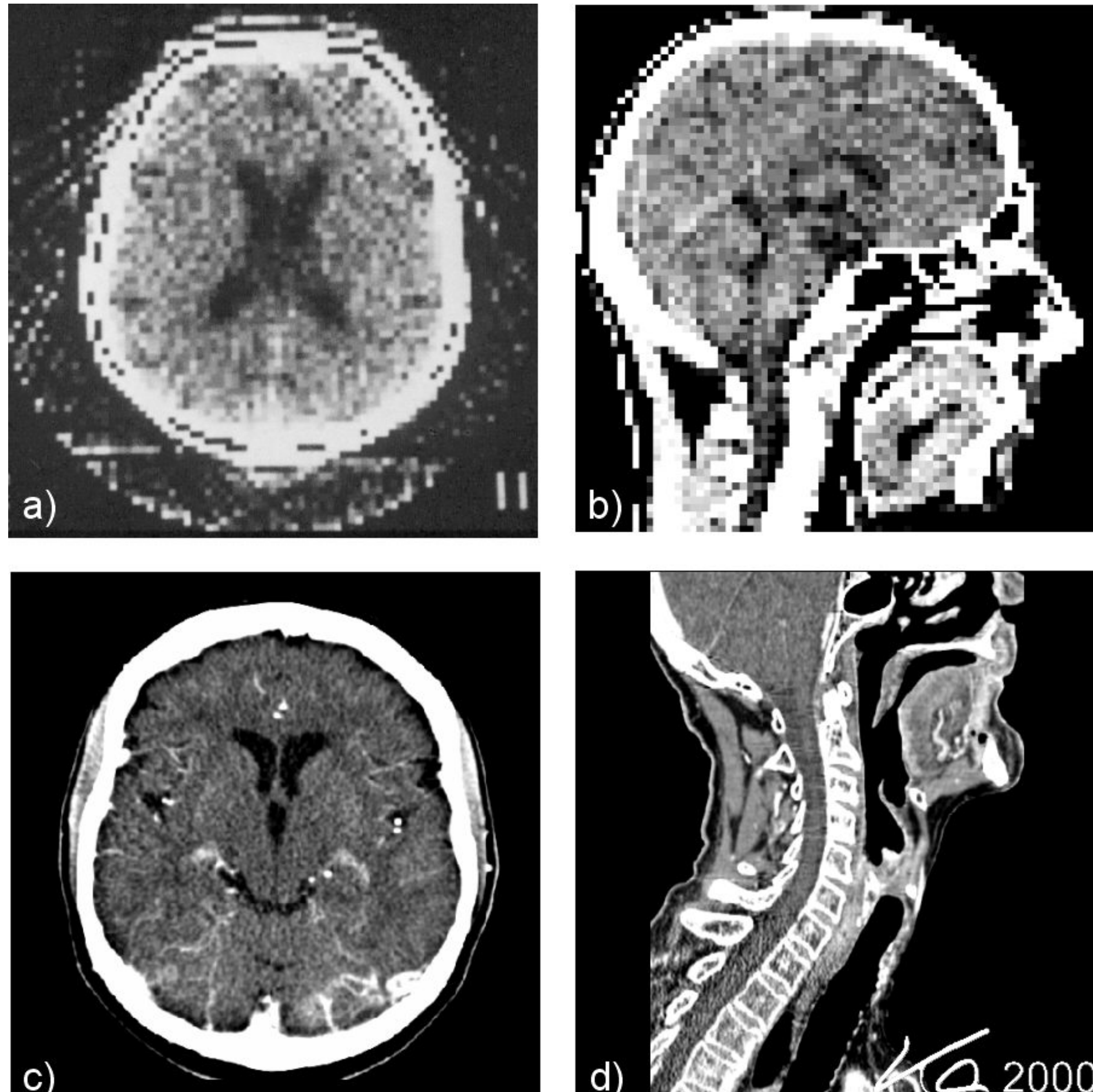
Overview

- Fundamentals of tomography
  - Physics & geometry
- Analytic formulation
  - Radon transform
  - Filtered back-projection
- Algebraic formulation

Examples of tomographic imaging

Computed (X-ray) Tomography (CT)





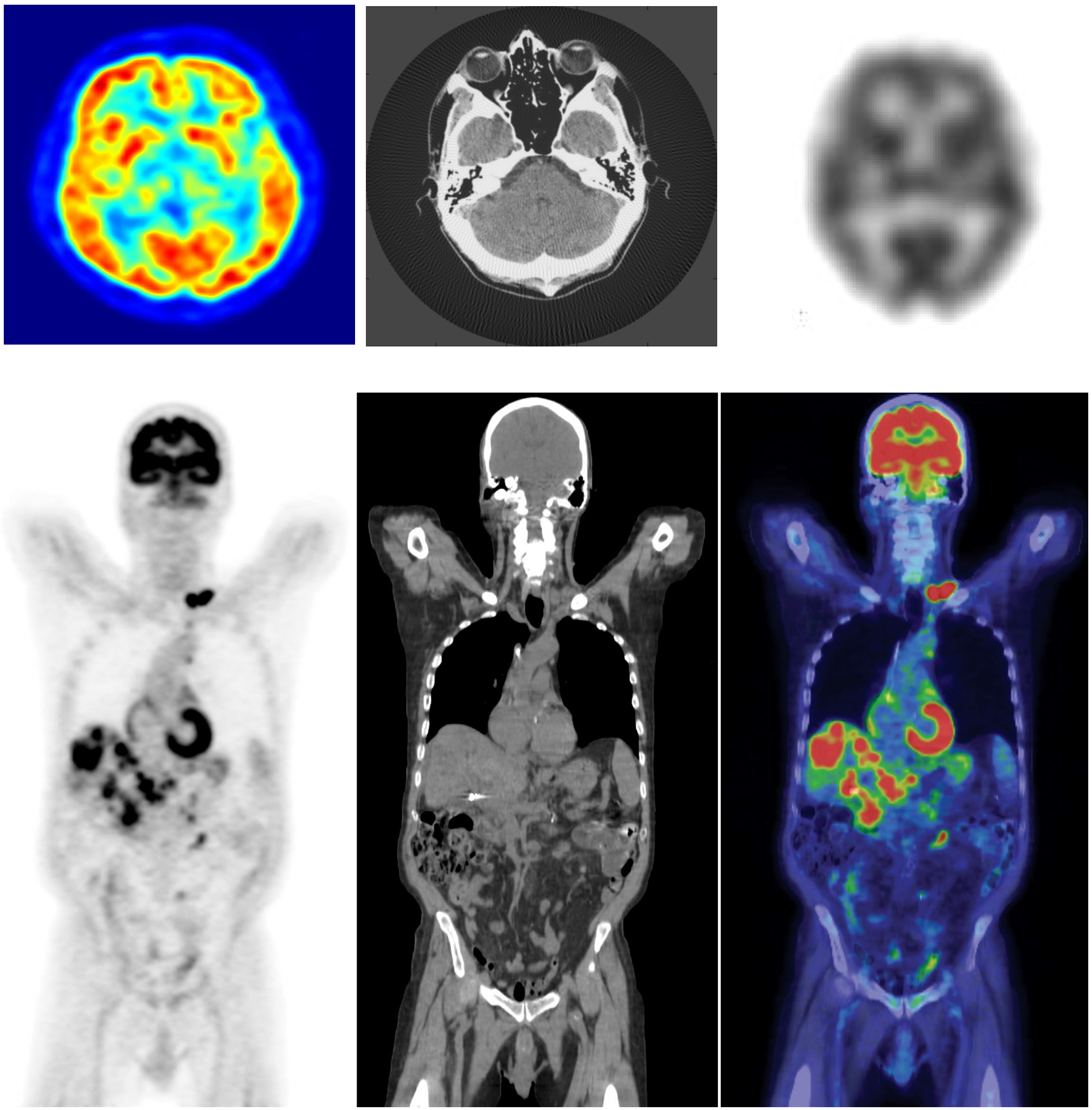
1974, 80x80 pixels

2000, 512x512 pixels, spiral CT

source: W. Kalender, Publicis, 3rd ed. 2011

Examples of tomographic imaging

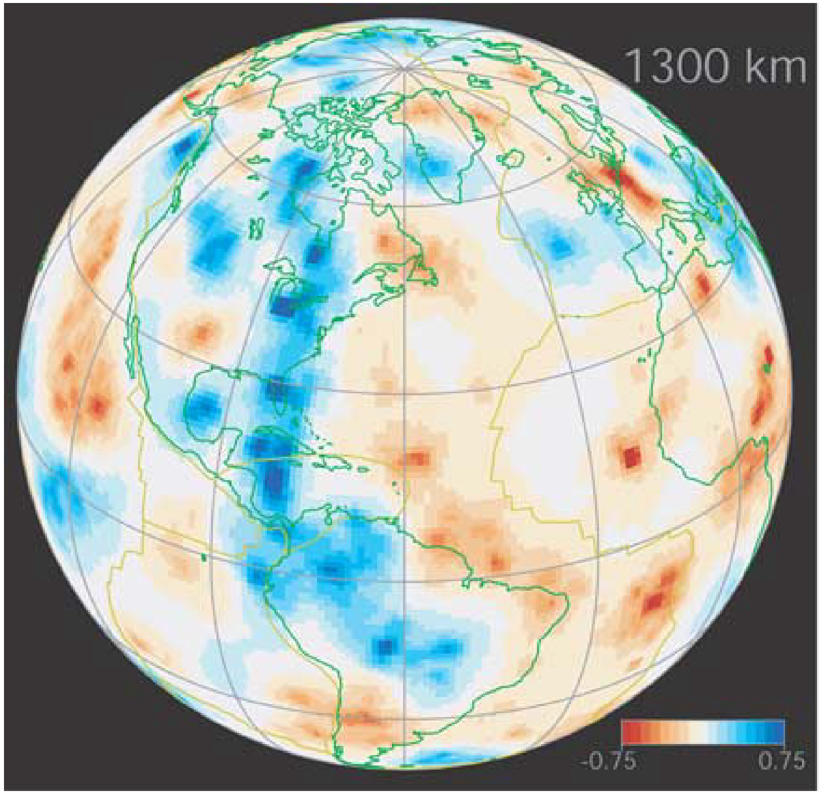
Positron emission tomography (PET) + CT

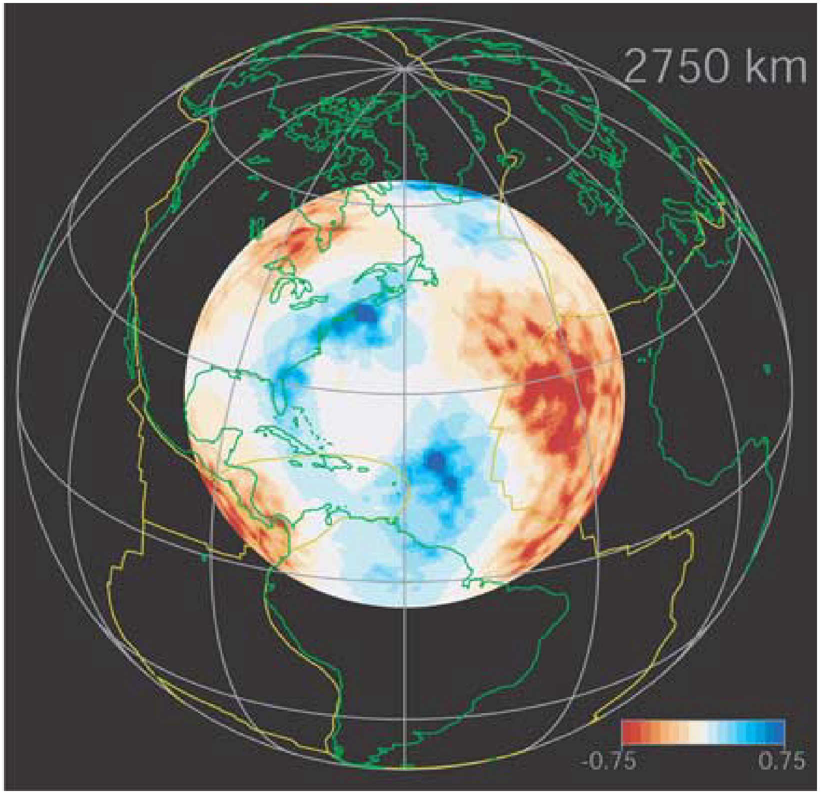


Single-Photon Emission Computed Tomography (SPECT)

Examples of tomographic imaging

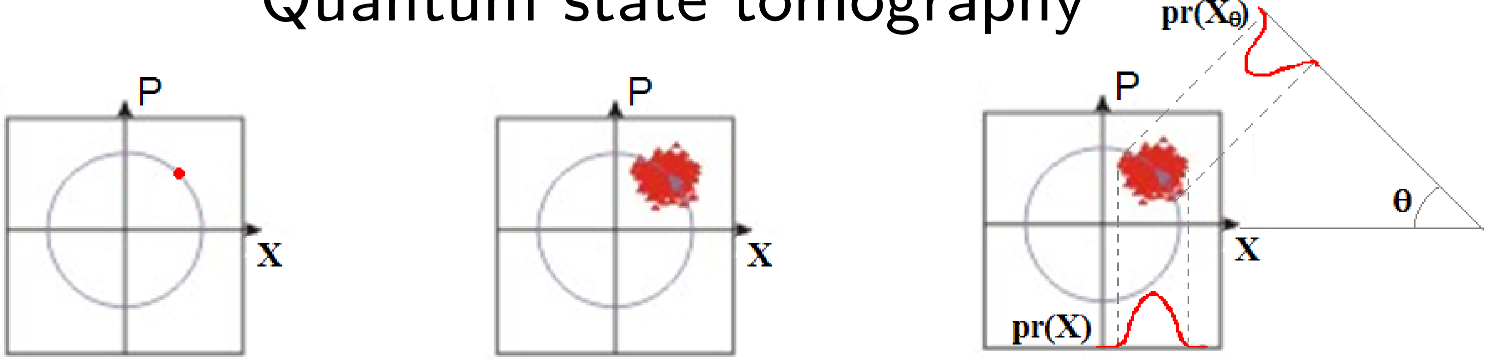
Seismic tomography





source: Sambridge et al. G3 Vol.4 Nr.3 (2003)

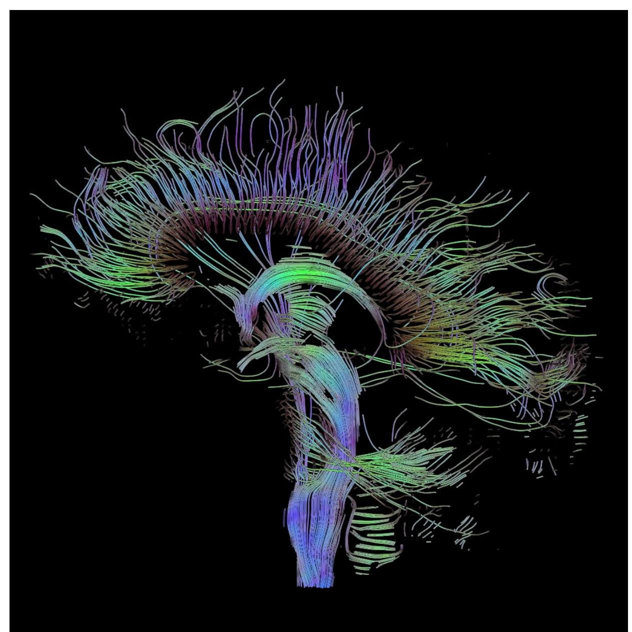
Quantum state tomography



Examples of tomographic imaging

Ultrasonography/tomography (US/UST)

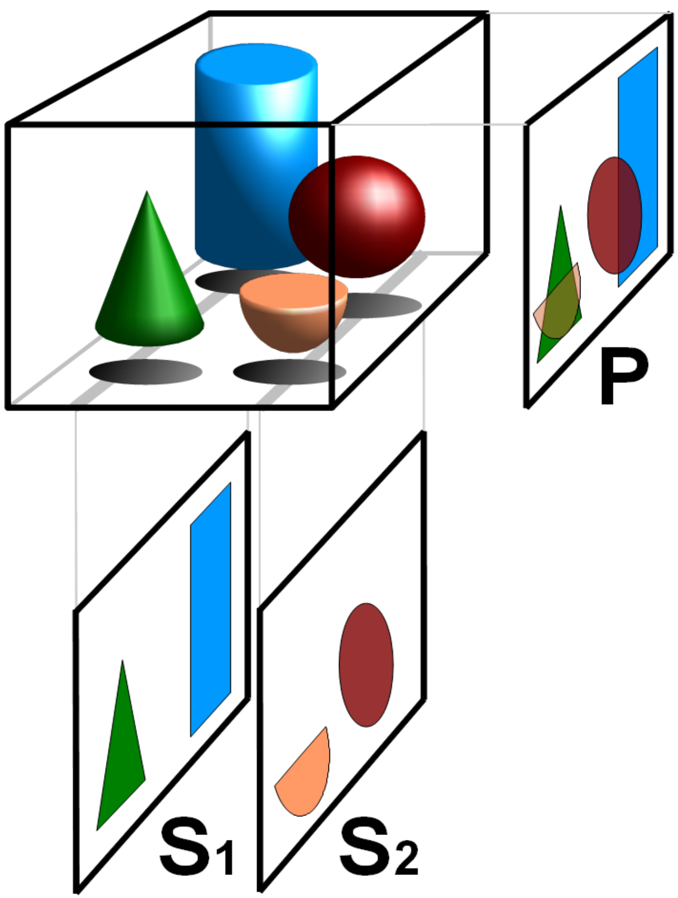
Magnetic resonance imaging/tomography (MRI/MRT)



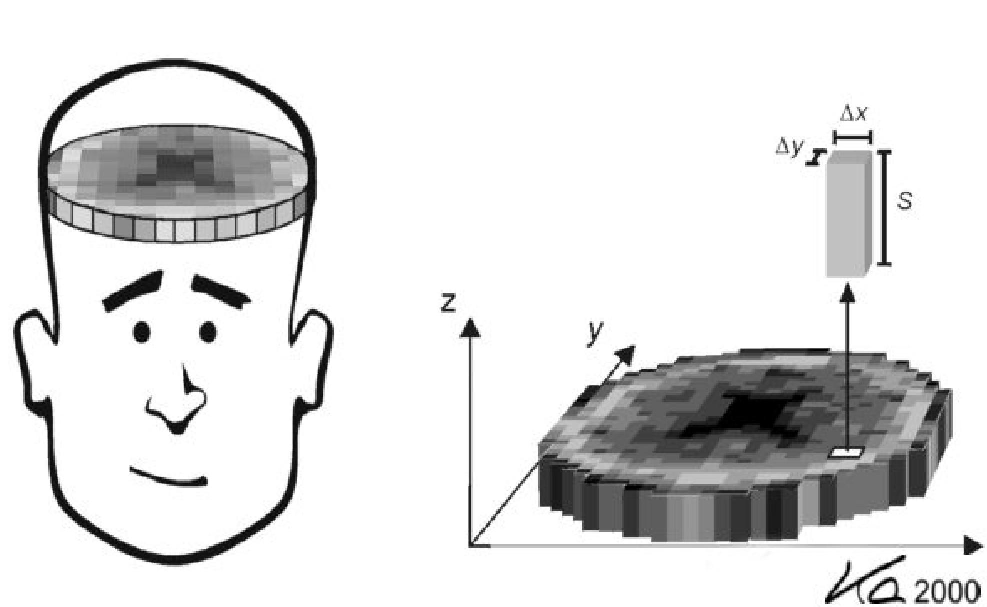
Not reconstruction from projections

Reconstructions from projections

Reconstruction of volume from projections

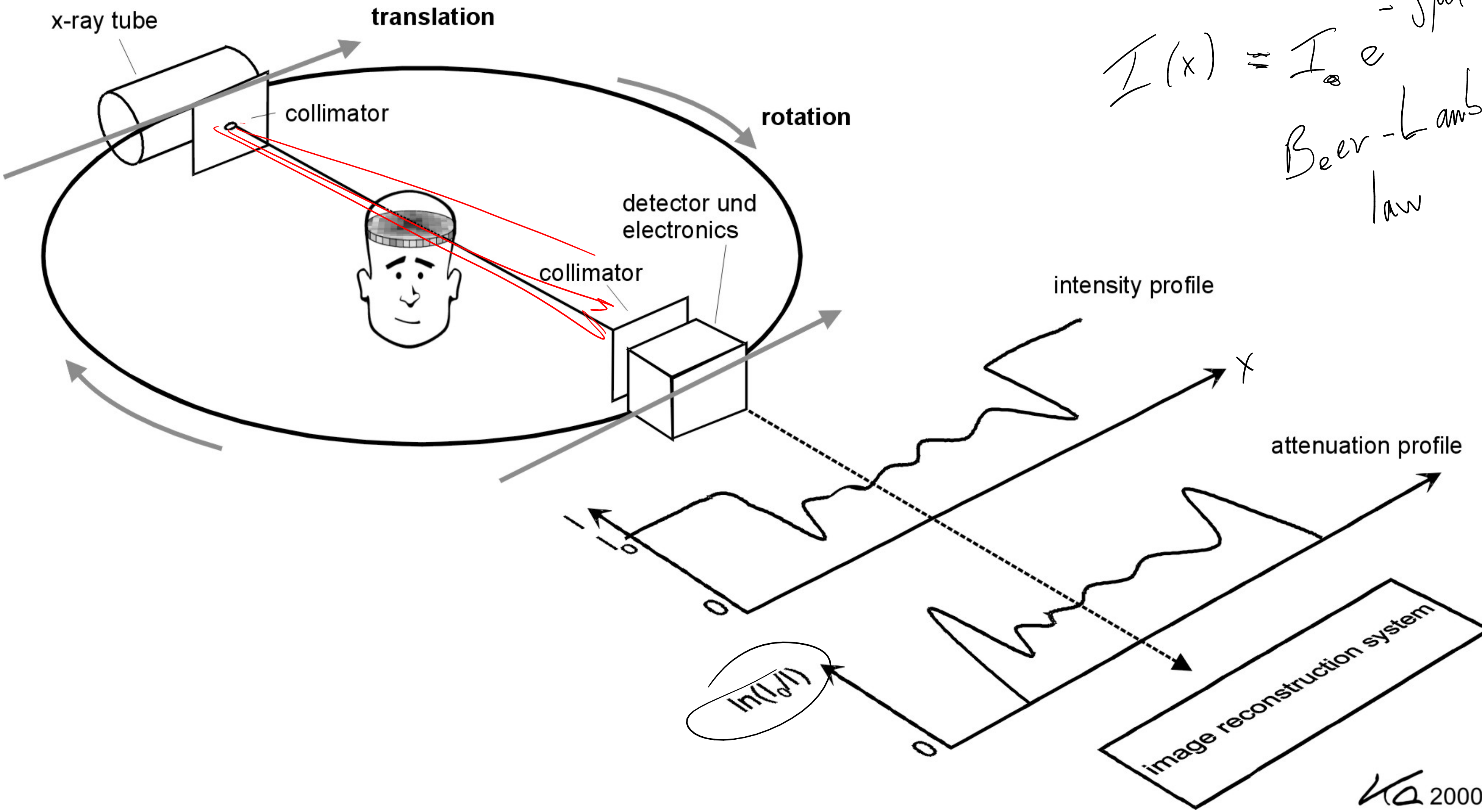


Digitization into voxels



source: W. Kalender, Publicis, 3rd ed. 2011

Principles of X-ray CT



thickness
↓

$$I(x) = I_0 e^{-\int \mu(s) ds}$$

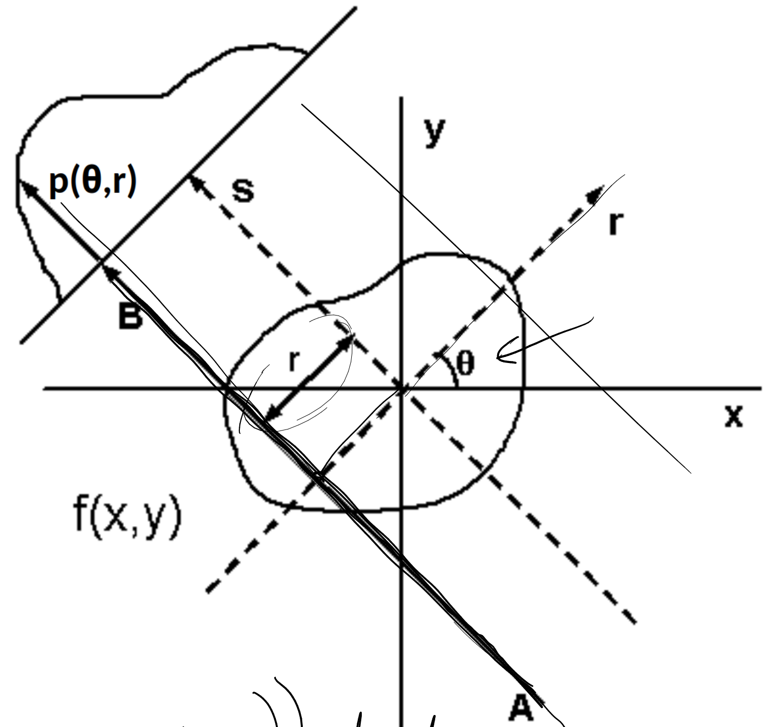
Beer-Lambert law

source: W. Kalender, Publicis, 3rd ed. 2011

Radon transform

Rotated coordinate system

Radon transform



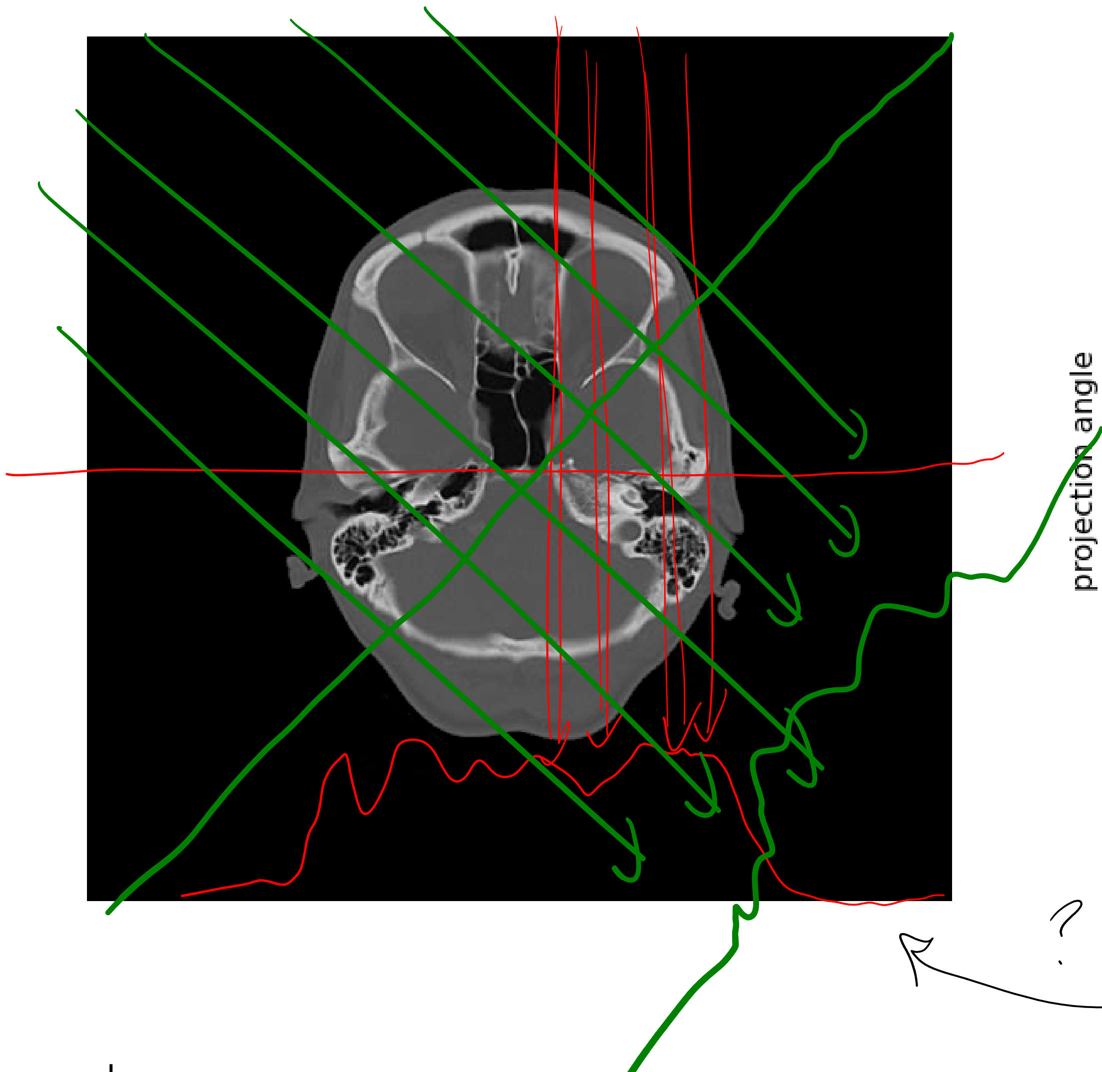
$$p(r, \theta) = \iint f(x,y)\, \delta(r - (x\cos\theta + y\sin\theta))\, dx\, dy$$

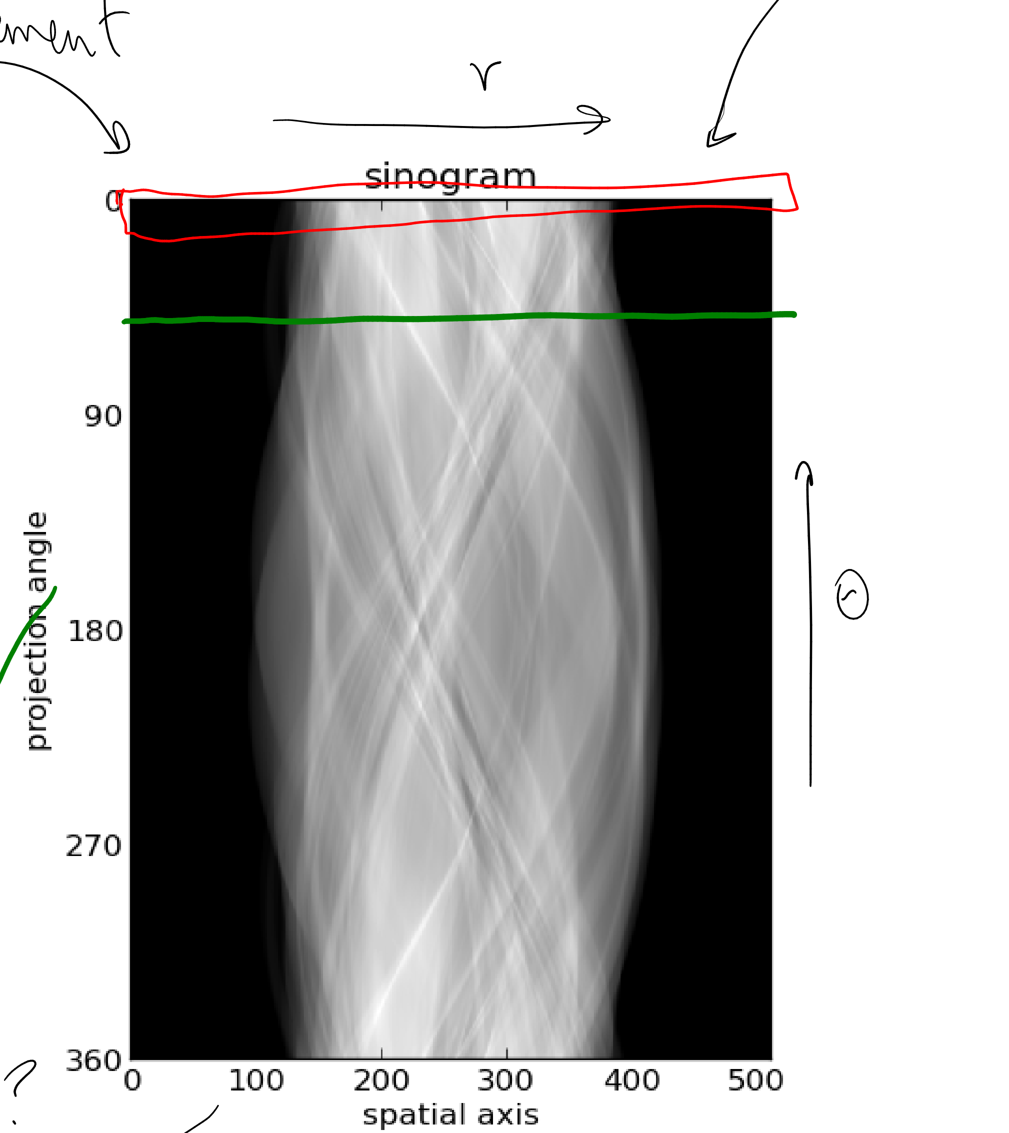
↑ can be negative

Problem: invert Radon transform

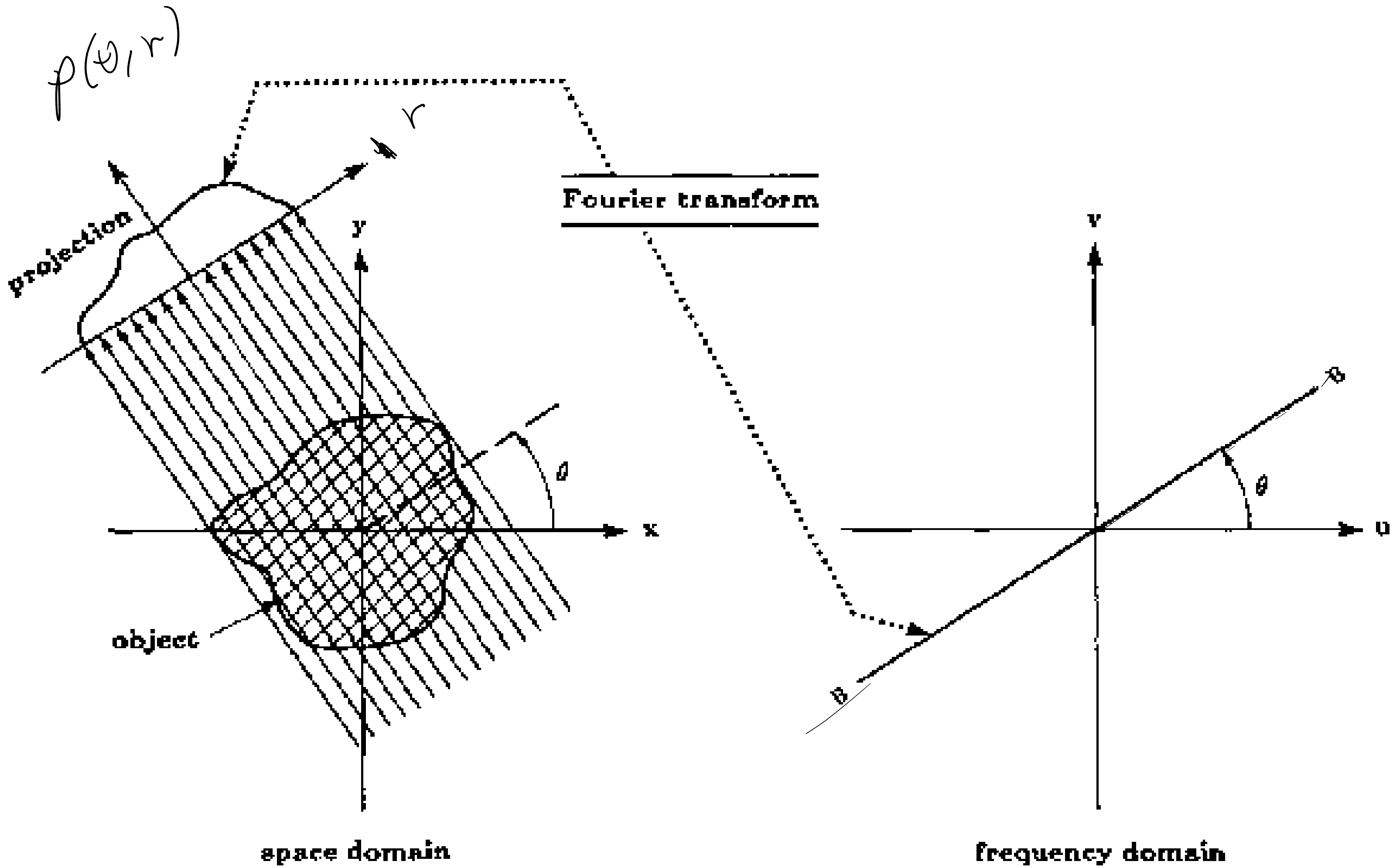
Sinogram

Representation of projection measured by a single detector line as a function of angle





The Fourier slice theorem



$\longleftarrow$ reciprocal variable $(x \Leftrightarrow u)$ $(r \Leftrightarrow s)$

$$\mathcal{F}_r\{p(\theta, r)\} = \int_{-\infty}^{\infty} p(r,\theta)\, e^{-2\pi i r s}\, dr$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\infty}^{\infty} f(x,y)\, \delta\big(r - (x\cos\theta + y\sin\theta)\big)\, e^{-2\pi i r s}\, dr\, dx\, dy$$

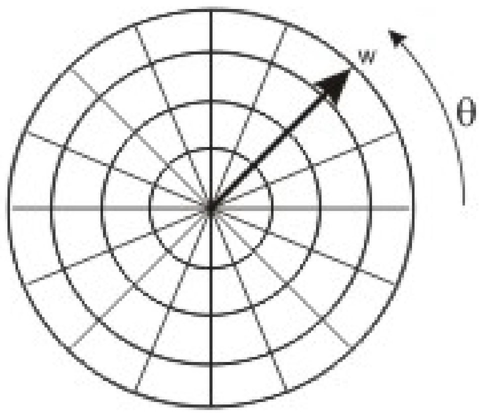
$$= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} f(x,y)\, e^{-2\pi i s (x\cos\theta + y\sin\theta)}\, dx\, dy$$

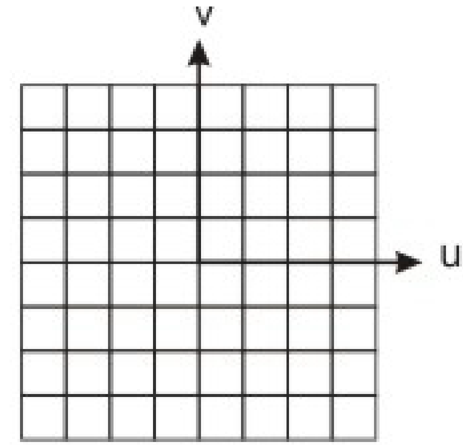
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i [\underbrace{x\, s\cos\theta}_{x \cdot u} + \underbrace{y\, s\sin\theta}_{y \cdot v}]}\, dx\, dy$$

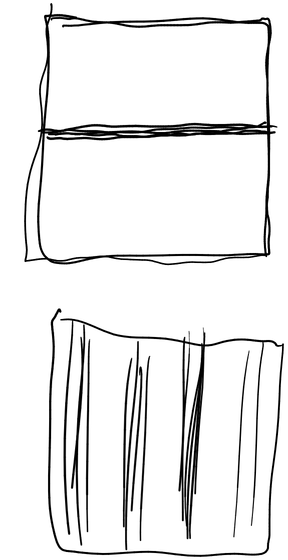
$$= F(u = s\cos\theta,\ v = s\sin\theta)$$

Frequency space sampling

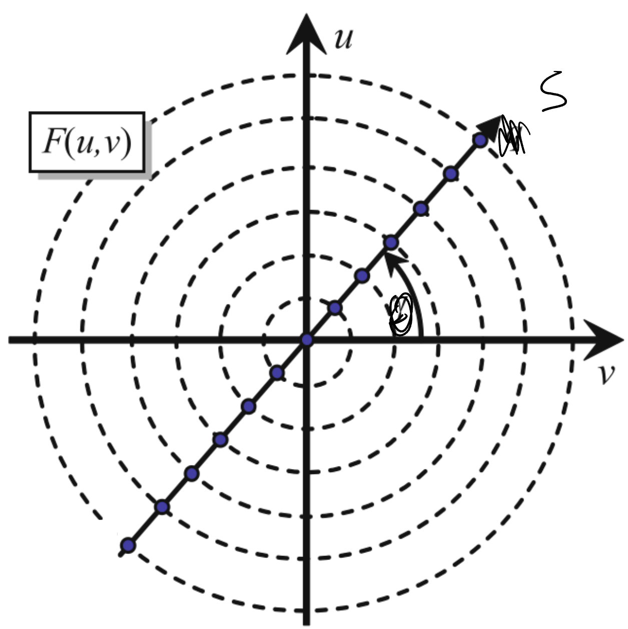
Change of sampling grid from polar to rectangular requires interpolation



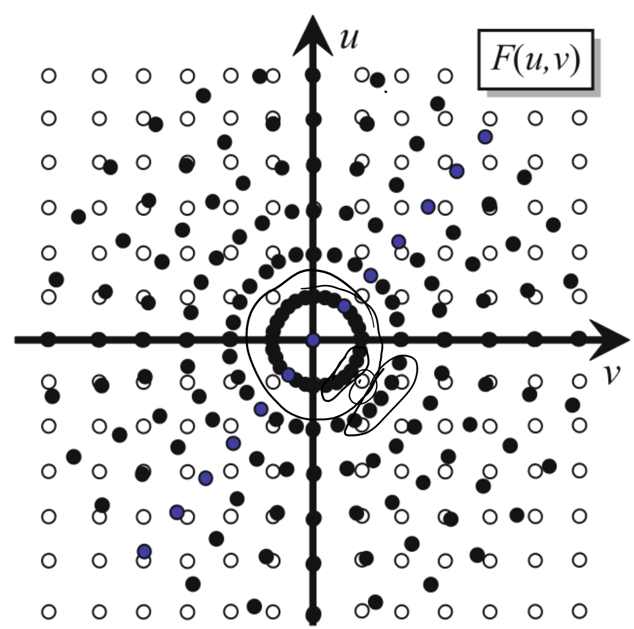




"grid rec"
regridding
Fourier space



a



b

Filtered back-projection

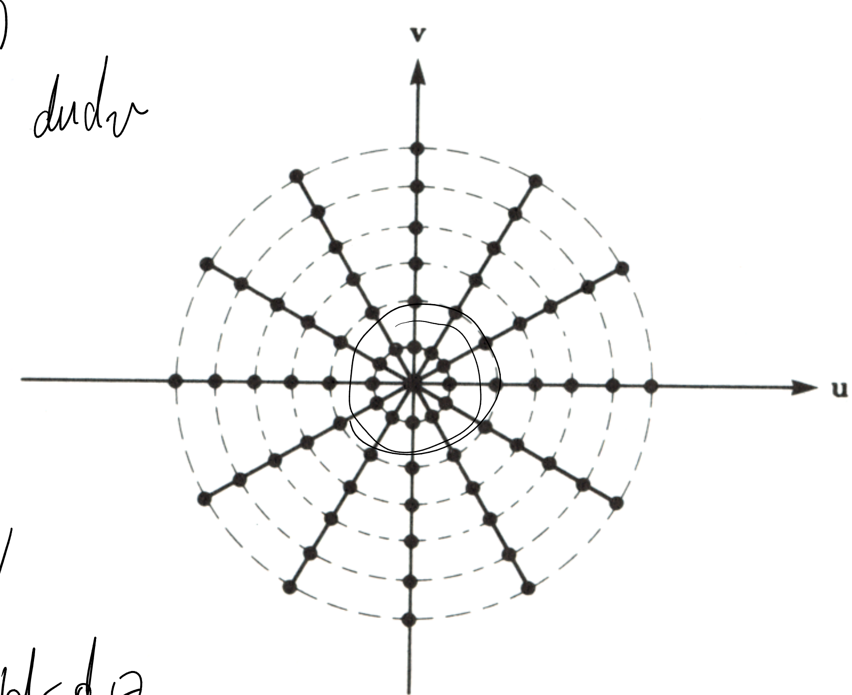
$$f(x,y) = \mathcal{F}^{-1}\{F(u,v)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i(ux+vy)}\, du\, dv$$

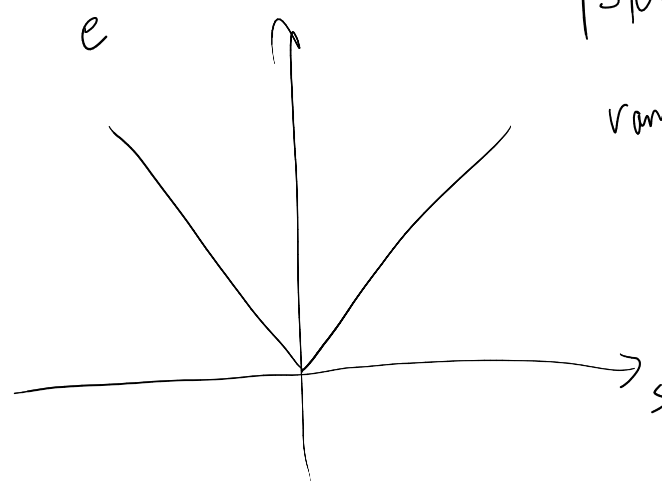
Polar coordinates $du\,dv \rightarrow s\,ds\,d\theta$

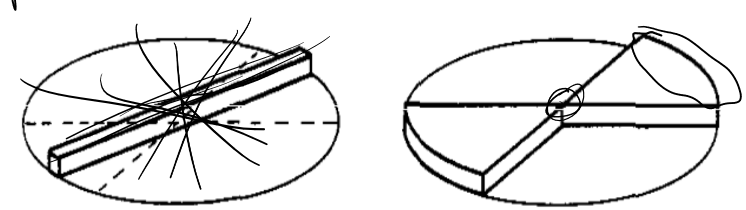
$$= \int_{0}^{\pi}\int_{-\infty}^{\infty} F(s\cos\theta, s\sin\theta)\, e^{2\pi i s(x\cos\theta + y\sin\theta)}\, |s|\, ds\, d\theta$$

ramp filter



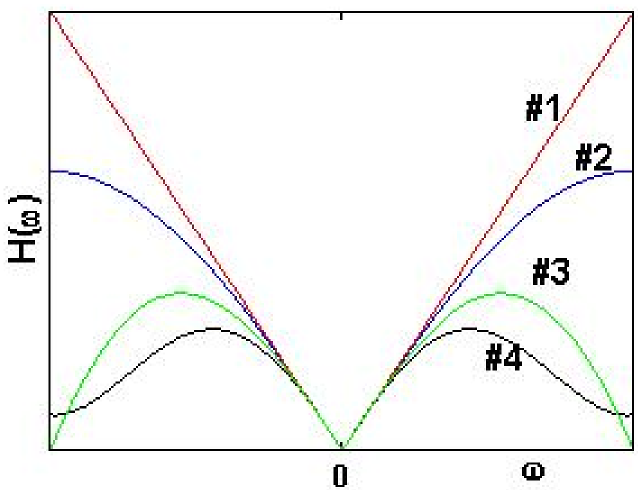
frequency domain





Filtered back-projection

- Filter can be tuned to achieve image enhancement
- Trade-off between noise and sharpness

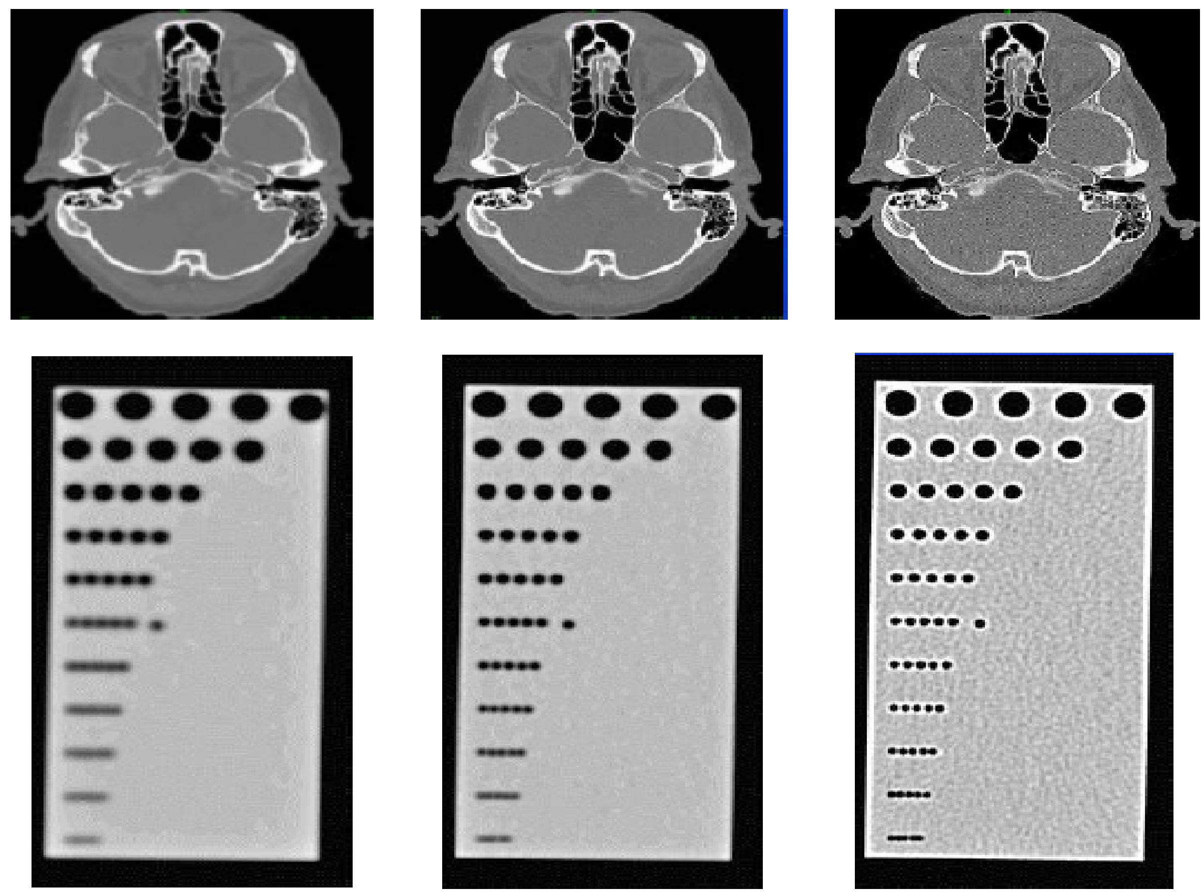


#1 ram-lak (ramp)

#2 Shepp-Logan

#3 cosine

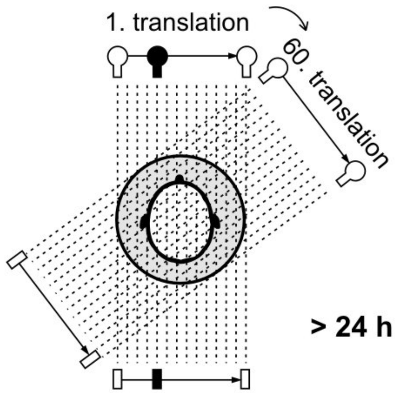
#4 Hamming



smoothing | standard | edge enhanced

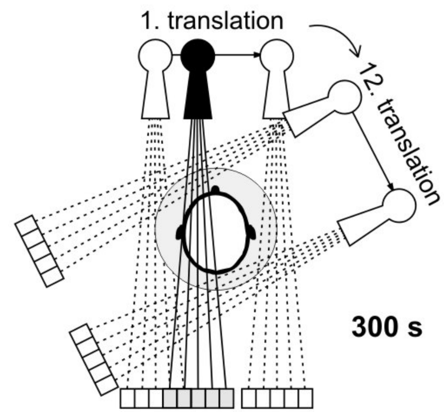
Geometries

pencil beam (1970)

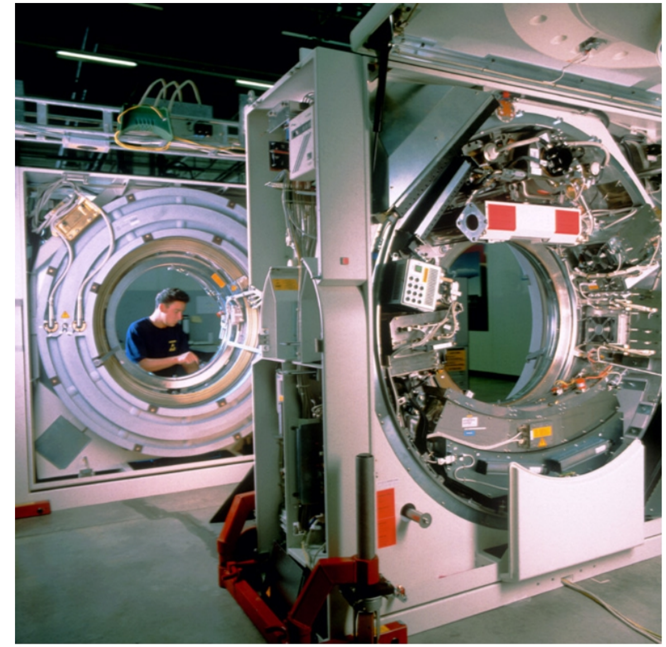


1^{st} generation: translation / rotation

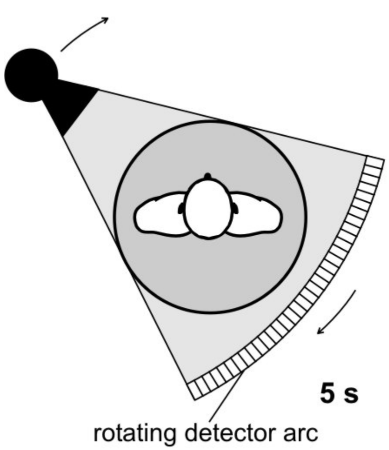
partial fan beam (1972)



2^{nd} generation: translation / rotation

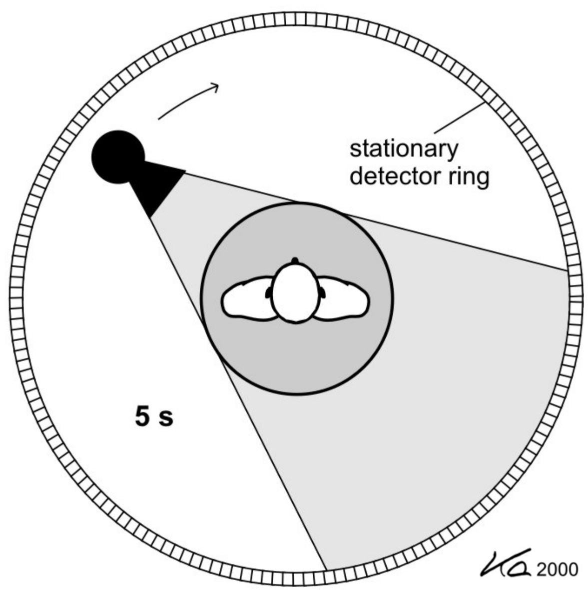


fan beam (1976)

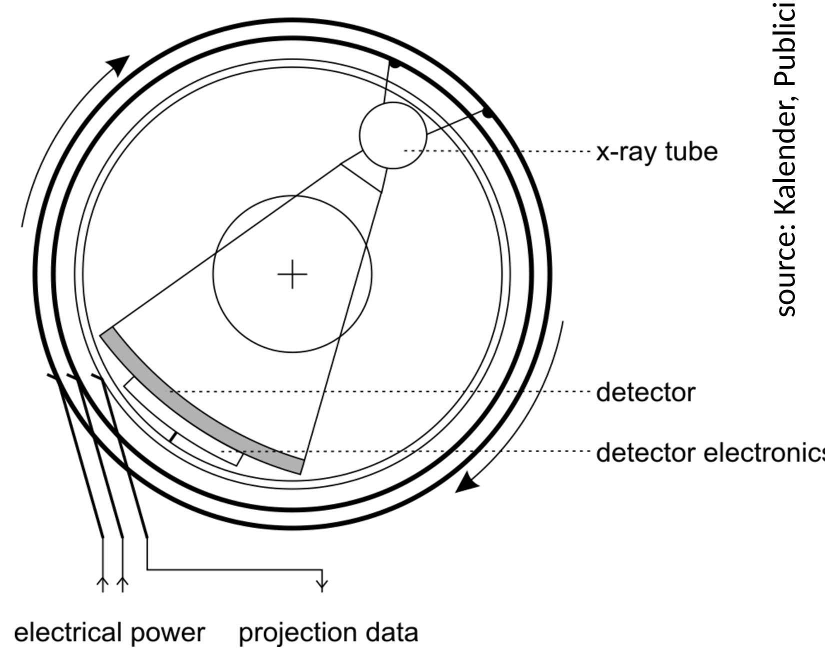


3^{rd} generation: continuous rotation

fan beam (1978)



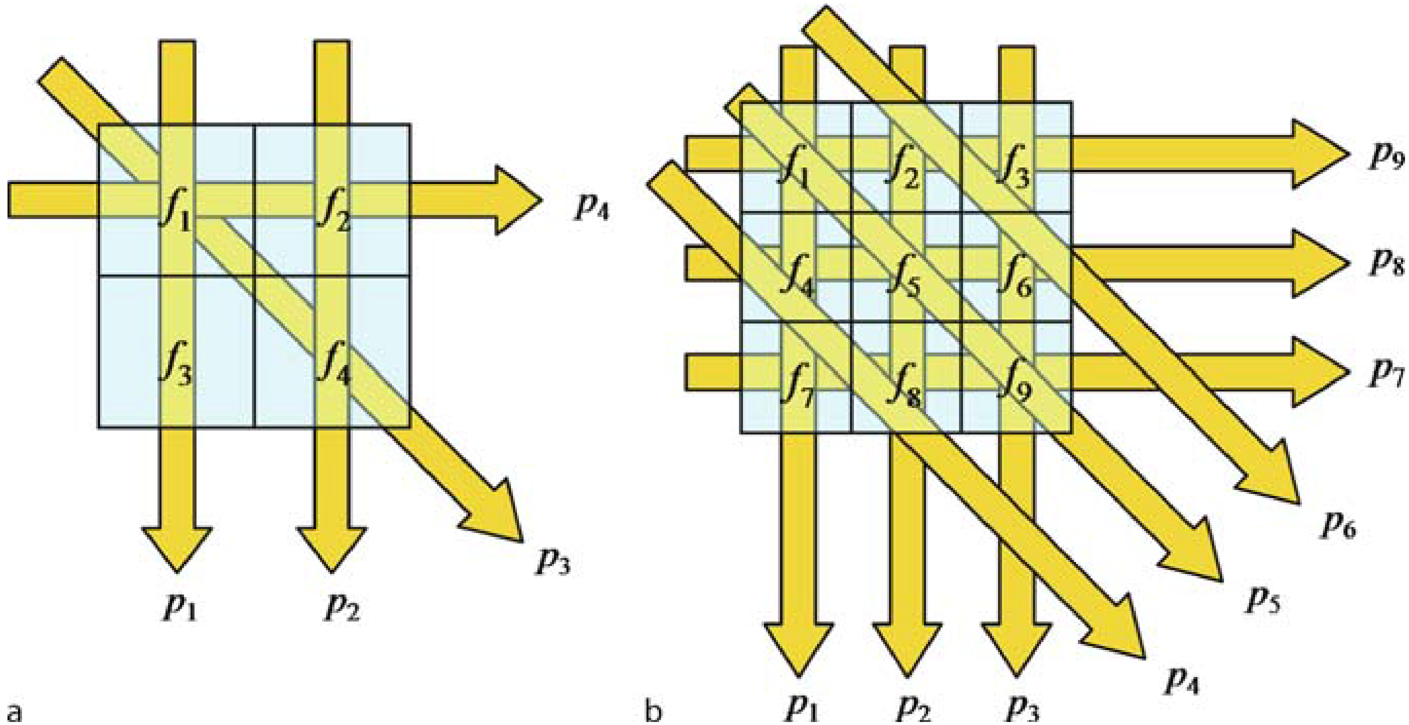
4^{th} generation: continuous rotation



source: Kalender, Publicis, 3rd ed. 2011

Algebraic formulation

Tomography can be formulated as a set of linear equations



$p_1 = f_1 + f_3$

$p_2 = f_2 + f_4$

$p_3 = f_1 + f_4$

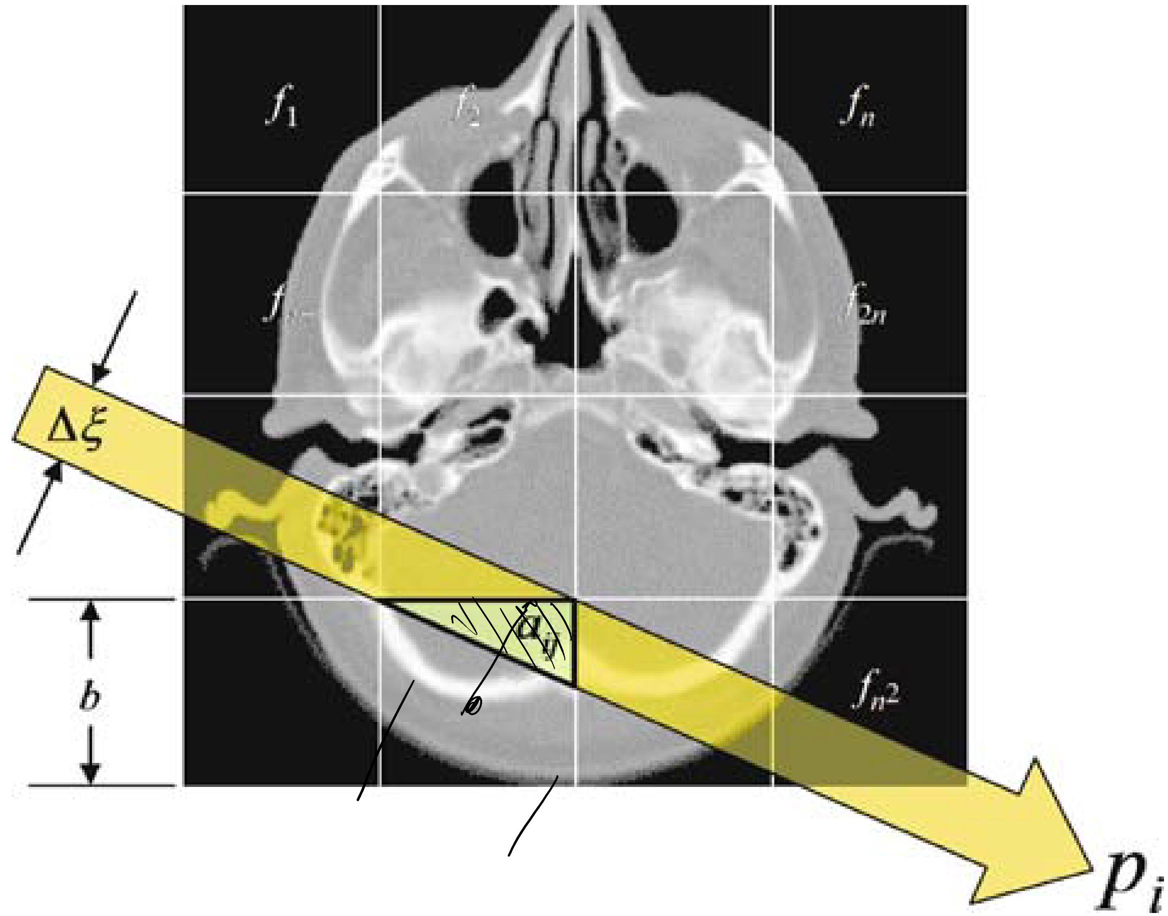
$p_4 = f_1 + f_2$

source: Buzug, Springer, 1st ed. 2008

Weighting coefficients

Weighting measures:

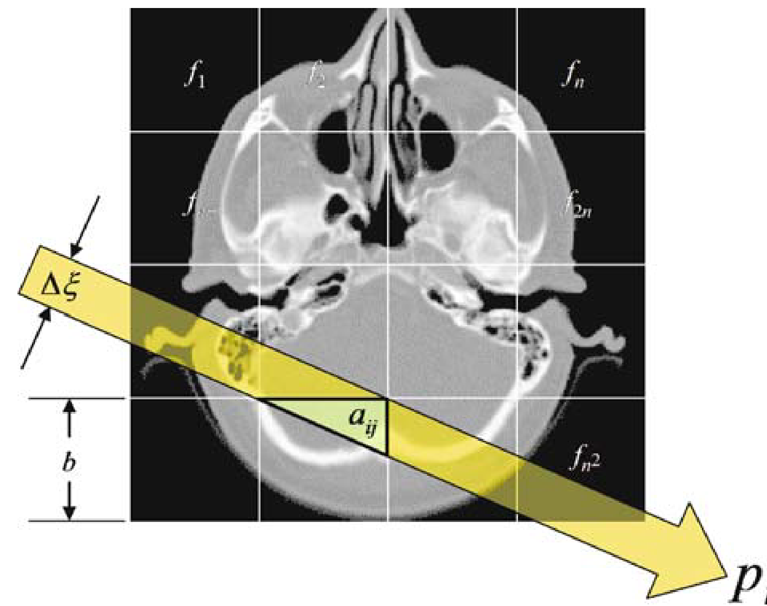
- Logic
- Area
- Path length
- Distance to pixel center

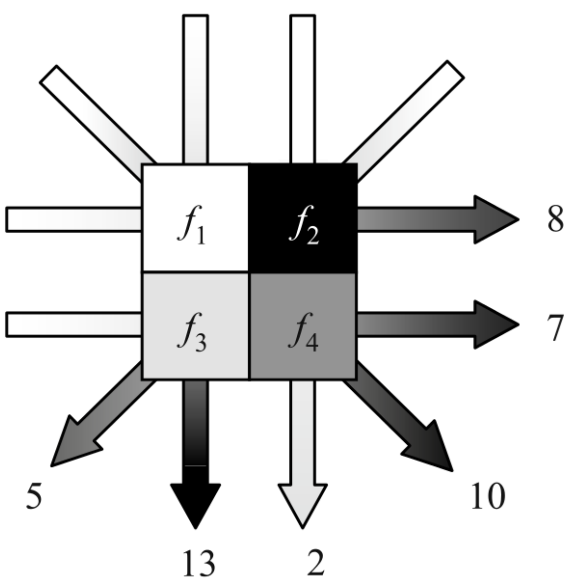


Differences in calculation effort, smoothness, noise sensitivity, ...

source: Buzug, Springer, 1st ed. 2008

System Matrix



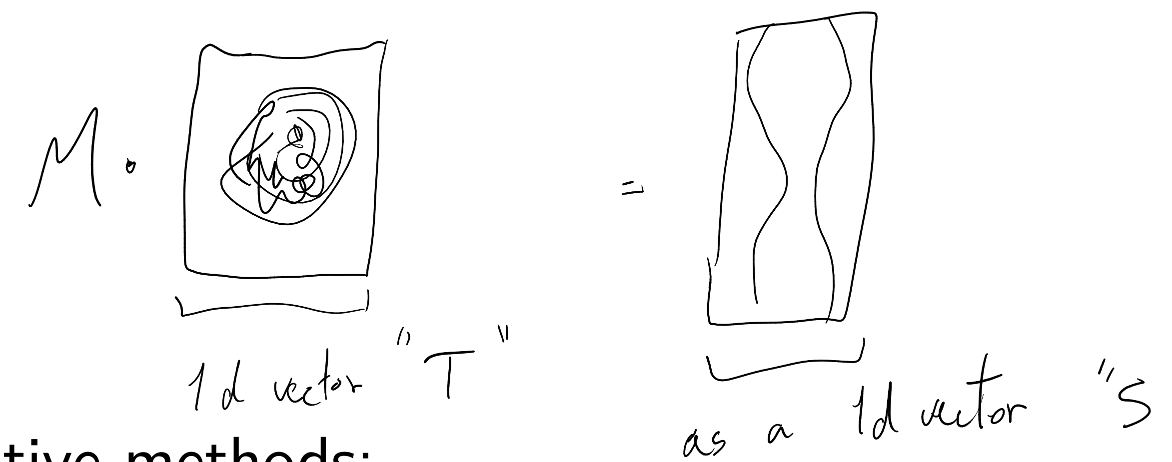


$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}$$

source: Buzug, Springer, 1st ed. 2008

Matrix (pseudo)-inversion

Tomographic reconstruction = linear system inversion



$M \cdot T = S$

$T = 10^6$ entries

$S = 10^6$

M : 1000000 x 1000000 matrix

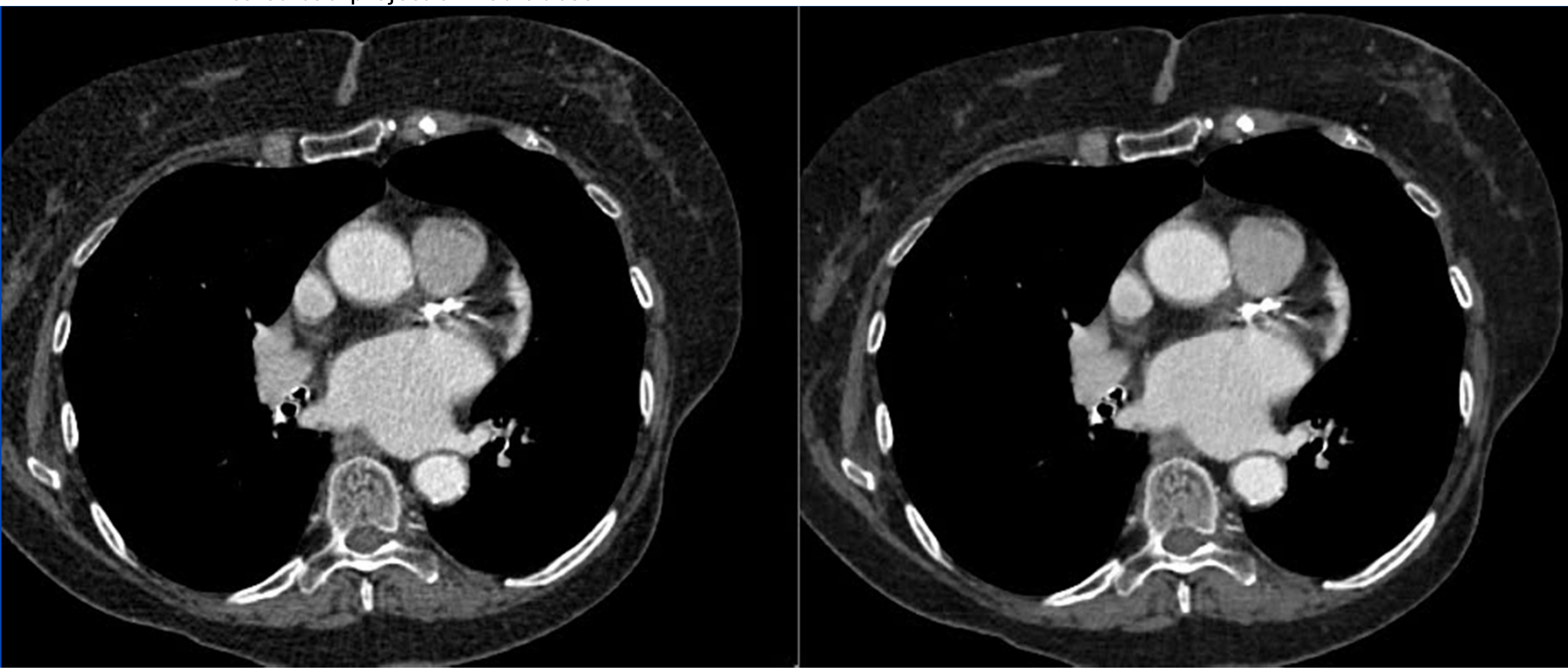
Iterative methods:

- ART Algebraic reconstruction technique
- SART Simultaneous algebraic reconstruction technique
- SIRT Simultaneous iterative reconstruction technique
- MART Multiplicative algebraic reconstruction technique
- MLEM Maximum likelihood expectation maximization
- OSEM Ordered subset expectation maximization
- ... and many, many more

FBP vs algebraic methods

FBP ART

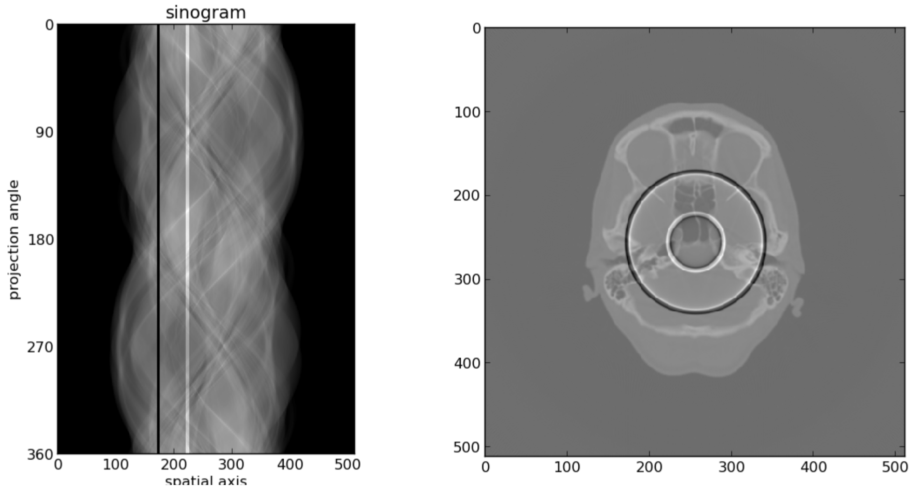
Filtered backprojection 100% dose | iterative 40% dose



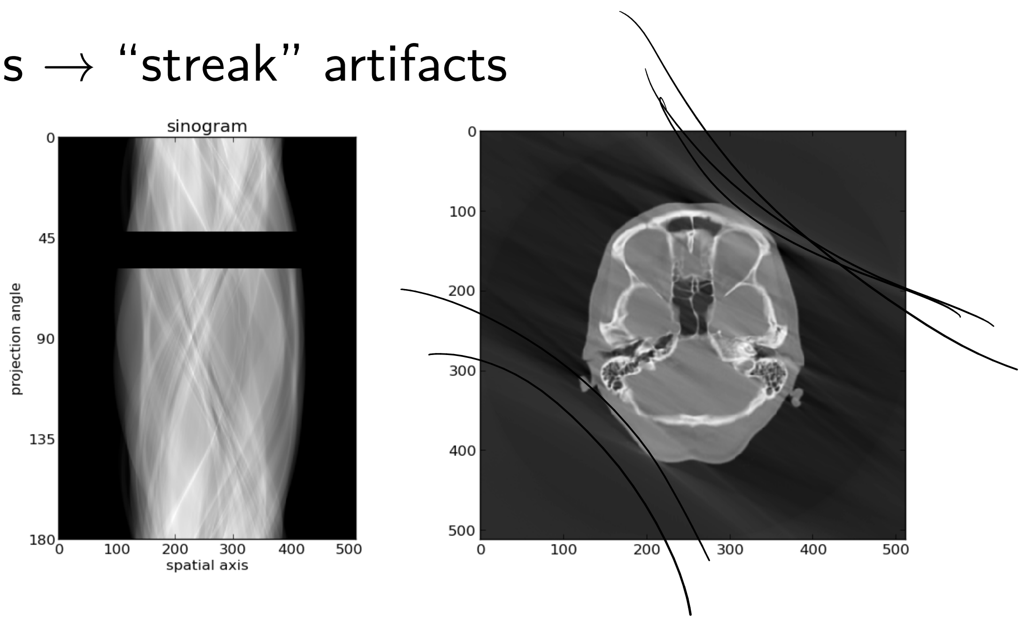
source: Kachelries, http://www.dkfz.de/en/medphysrad/workinggroups/ct/ct_conference_contributions/BasicsOfCTImageReconstruction_Part2.pdf

Artifacts

Detector imperfections → ring artifacts



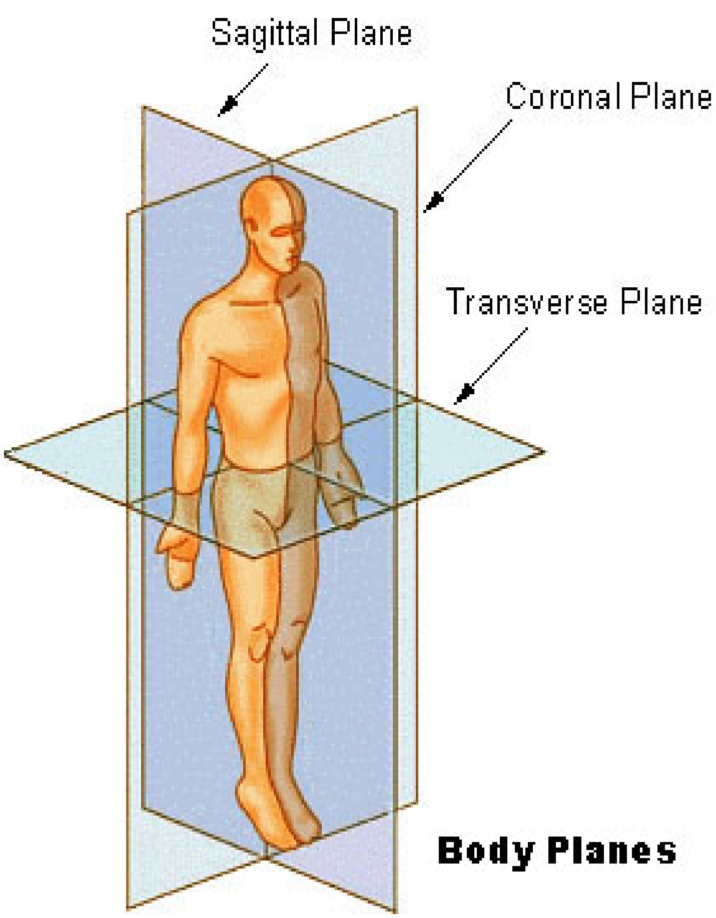
Missing projections → "streak" artifacts

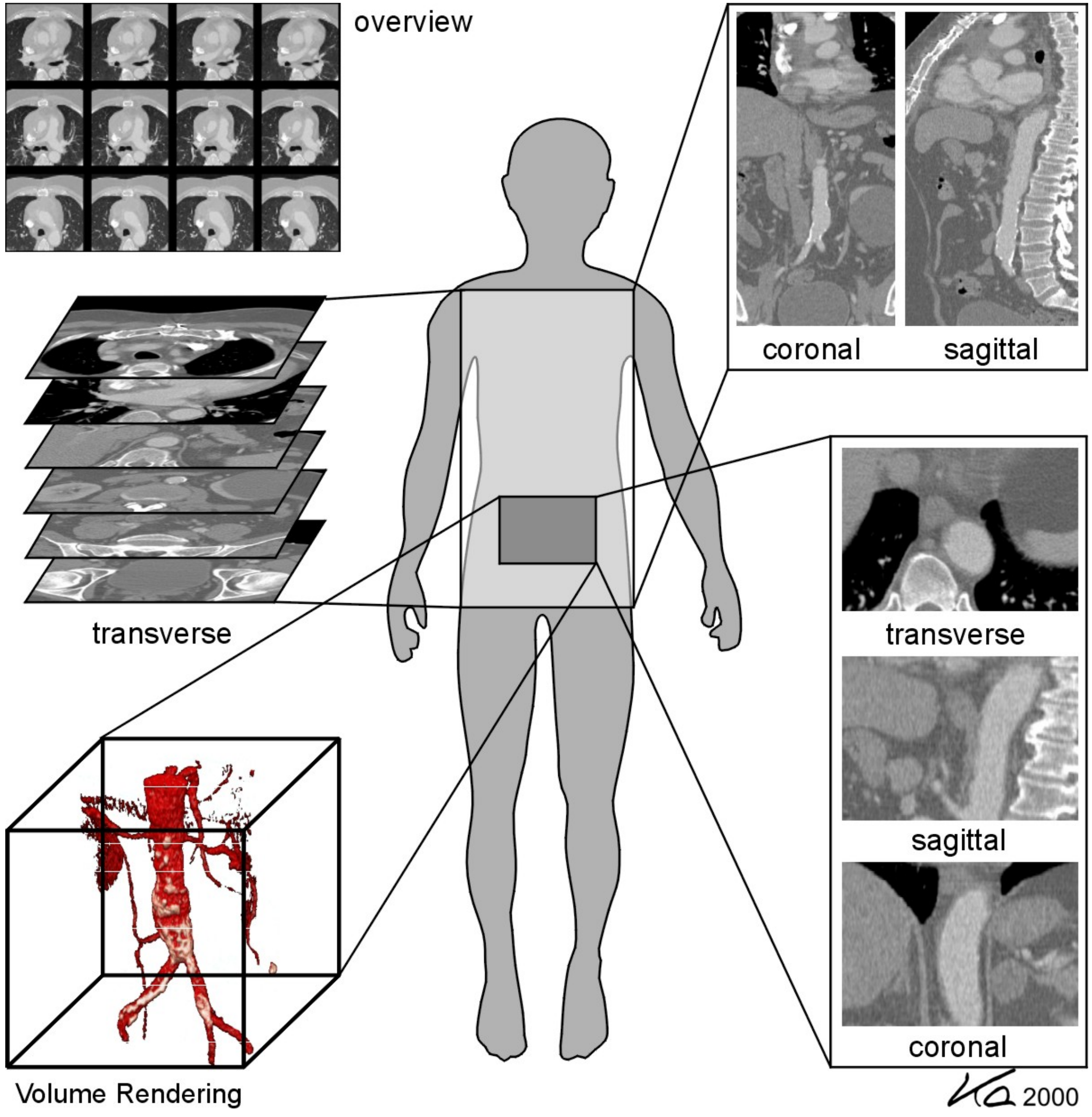


Also: sample motion, beam hardening, ...

polychromatic

Tomographic Display

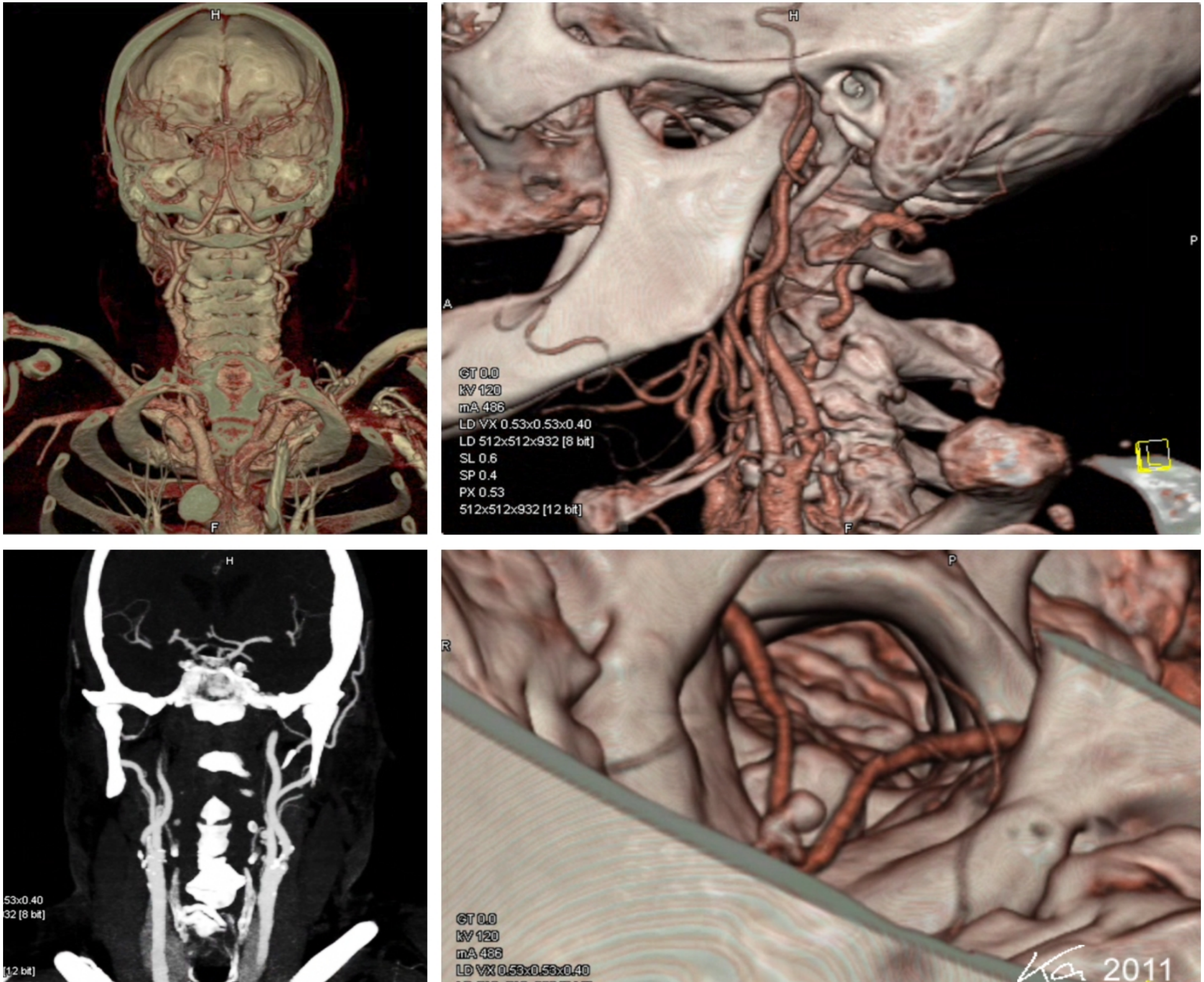




source: http://wikipedia.org

source: W. Kalender, Publicis, 3rd ed. 2011

Volume rendering display



source: Kalender, Publicis, 3rd ed. 2011

Summary

- Computed tomography: reconstruction from projections
- Analytic approach:
  - Projections and tomographic slices are related by the Fourier slice theorem
  - Standard algorithm uses filtered back-projection
- Algebraic approach:
  - Tomography as a system of linear equations
  - Iterative methods are used for large matrix inversions
  - More powerful but computationally more costly
- Imperfect data leads to artifacts