DETERMINANTE

$A \in M_n(\mathbb{K})$

$A \rightsquigarrow \underline{\underline{\det A}} \in \mathbb{K}$ $\qquad A = (a_{ij})$

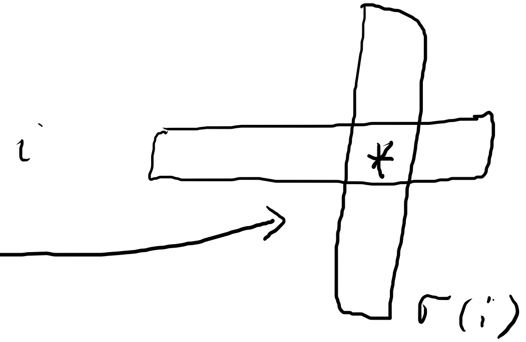
Def

$$\det A \overset{\text{def}}{=} \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$n!$ addendi

$\{1, 2, 3, \dots, n\} \circlearrowleft \sigma$

$\underline{\underline{a_{i\sigma(i)}}} \in A^{(i)} \cap A_{(\sigma(i))}$



$\boxed{n=1}$ $A = (a)$ $\Sigma_1 = \{id\}$

$\det A = a$

$n=2$ $\Sigma_2 = \{\underset{+1}{id}, \underset{-1}{(1\ 2)}\}$

$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underset{\binom{\sigma = id}{+}}{a_{11} a_{22}} - \underset{\binom{\sigma = (1\ 2)}{-}}{a_{12} a_{21}}$

$\det \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} = 12 + 2 = 14$

$\shortparallel$

$\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix}$

$\det A = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$

$\boxed{\det A = |A|}$

$\begin{vmatrix} 2i & 3+i \\ -2+i & 1 \end{vmatrix} =$

$= 2i - (3+i)(-2+i) =$

$= 2i + 6 - 3i + 2i + 1 =$

$= 7 + i.$

$\underline{n=3}$

$$\begin{array}{ccccccc} \Sigma_3 = \{ & id, & (1\ 2), & (1\ 3), & (2\ 3), & (1\ 2\ 3), & (1\ 3\ 2)\ \} \\ & 1 & -1 & -1 & -1 & +1 & 1 \end{array}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underline{a_{11}a_{22}a_{33}} - \underline{a_{12}a_{21}a_{33}} - \underline{a_{13}a_{22}a_{31}} -$$

$$- \underline{a_{11}a_{23}a_{32}} + \underline{a_{12}a_{23}a_{31}} + \underline{a_{13}a_{21}a_{32}}$$

Regola di $\underline{\underline{\text{SARRUS}}}$
(3×3)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{pmatrix}$$

$\searrow\ +$ $\qquad$ $-\ \swarrow$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} = 0 + 2 - 2 - 0 - 1 - 0 = -1$$

Proprietà dei determinanti

Prop. $\det A = \det {}^tA \quad , \quad \forall A \in M_n(\mathbb{K})$

Dim. $A = (a_{ij}) \quad , \quad {}^tA = (a'_{ij})$

$a'_{ij} = a_{ji}$

$\sigma \in \Sigma_n \rightsquigarrow \sigma^{-1}$

$\sigma\sigma^{-1} = id$

$\operatorname{sgn}(\sigma\sigma^{-1}) = \operatorname{sgn}(id) = 1$

$\operatorname{sgn}\sigma \cdot \operatorname{sgn}(\sigma^{-1}) = 1$

$\Downarrow$

$\operatorname{sgn}\sigma = \operatorname{sgn}(\sigma^{-1})$

$$\det {}^tA = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma)\, a'_{1\sigma(1)} \cdots a'_{n\sigma(n)} = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

$$a_{\sigma(1)1} \cdots a_{\sigma(n)n} = a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$$

Es $n = 3 \quad , \quad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$a_{31}\, a_{12}\, a_{23} = a_{12}\, a_{23}\, a_{31}$

$$= \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma)\, a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)} = \sum_{\sigma^{-1}\in\Sigma_n} \operatorname{sgn}(\sigma^{-1})\, a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)} =$$

$$= \det A$$

$\mathbb{R}^2$ $\qquad \mathcal{E} = (e_1, e_2)$, $\qquad \mathcal{V} = \Bigg(\underbrace{v_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{Componenti rispetto alla base } \mathcal{E}} \Bigg)$

base canonica

Matrice del cambiamento di base : A

$$id : \underset{\mathcal{V}}{\mathbb{R}^2} \longrightarrow \underset{\mathcal{E}}{\mathbb{R}^2}$$

$$\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$A = M^{\mathcal{E}}_{\mathcal{V}}(id) = \begin{array}{c} \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \\ \;\; v_1 \quad\;\; v_2 \end{array}$$

$$\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$v_1 = 1 \cdot v_1 + 0\, v_2 \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ componenti di v_1 nella base $\mathcal{V}$

$v_2 = 0 \cdot v_1 + 1\, v_2 \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ " " v_2 " " $\mathcal{V}$

$u = x v_1 + y v_2$

$\underline{\underline{u}}$ ha componenti $\begin{pmatrix} x \\ y \end{pmatrix}^{\mathcal{V}}$ nella base $\mathcal{V}$ $\longrightarrow$ $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+y \\ -2x+y \end{pmatrix}^{\mathcal{E}}$

comp di $\underline{\underline{u}}$ nella base $\mathcal{E}$

$$u = (3x+y)\, e_1 + (-2x+y)\, e_2 = x v_1 + y v_2$$

$$u = \begin{pmatrix} 4 \\ -2 \end{pmatrix}^{\mathcal{V}} = \underbrace{\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}}_{\mathcal{E}} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \end{pmatrix}^{\mathcal{E}}$$

$$\underset{\mathcal{E}}{\mathrm{id}: \mathbb{R}^2} \longrightarrow \underset{\mathcal{V}}{\mathbb{R}^2}$$ è rappresentata dalla matrice $\underline{\underline{A^{-1}}}$

$$\begin{pmatrix} 3 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{5}{3} & \frac{2}{3} & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{5} & \frac{3}{5} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

$$w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}^{\mathcal{E}} = A^{-1} \underset{\mathcal{V}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \frac{1}{5} \underset{\mathcal{V}}{\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix}^{\mathcal{V}}$$

$$f : \underset{\mathcal{E}}{\mathbb{R}^2} \longrightarrow \underset{\mathcal{E}}{\mathbb{R}^2} \qquad f = L_T \quad , \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix}^{\mathcal{E}} = T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x \end{pmatrix}^{\mathcal{E}}$$

$$\mathcal{V} = (v_1 , v_2) \qquad v_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} , \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f : \underset{\mathcal{V}}{\mathbb{R}^2} \longrightarrow \underset{\mathcal{V}}{\mathbb{R}^2}$$

$$\begin{tikzcd} \underset{\mathcal{V}}{X} \arrow[r] \arrow[rrr, bend right] & \underset{\mathcal{E}}{AX} \arrow[r, "f"] & \underset{\mathcal{E}}{TAX} \arrow[r] & \underset{\mathcal{V}}{\underbrace{A^{-1}TA}} X \end{tikzcd}$$

$$M_{\mathcal{V}}^{\mathcal{V}}(f) = A^{-1}TA$$

$$A^{-1}TA = A^{-1}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} = A^{-1}\begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} =$$

$$= \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 19 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{19}{5} & \frac{3}{5} \end{pmatrix} = M^{\mathcal{V}}_{\mathcal{V}}(f)$$

In generale

$$f : V \longrightarrow V \qquad \dim V = n$$

$$\mathcal{V} \qquad \mathcal{V} \qquad \mathcal{V} = (v_1, \dots, v_n)$$

$$\mathcal{V}' \qquad \mathcal{V}' \qquad \mathcal{V}' = \underbrace{(v_1', \dots, v_n')}$$

$$T = M^{\mathcal{V}}_{\mathcal{V}}(f)$$

espresso nella base $\mathcal{V}$

$$A = \begin{pmatrix} v_1', \dots, v_n' \\ \text{colonne} \end{pmatrix} = M^{\mathcal{V}}_{\mathcal{V}'}(id) \in GL_n(\mathbb{K})$$

$$M^{\mathcal{V}'}_{\mathcal{V}}(id) = \underline{\underline{A^{-1}}}$$

$$\begin{array}{ccccccc} X & \longrightarrow & AX & \xrightarrow{f} & TAX & \longrightarrow & \underline{A^{-1}TAX} \\ \mathcal{V}' & & \mathcal{V} & & \mathcal{V} & & \mathcal{V}' \end{array}$$

$$\underline{M^{\mathcal{V}'}_{\mathcal{V}'}(f) = A^{-1}TA}$$

Def Diciamo che $T, T' \in M_n(\mathbb{K})$ sono simili

se $\exists\ A \in GL_n(\mathbb{K})$ t.c.

$$\boxed{T' = A^{-1} T A}$$

Similitudine tra matrici $n \times n$

rel. d'equiv. (*)