27 Novembre

$$f(x) = \frac{[x]+1}{[x]^4+[x]+1}$$ definita su $\mathbb{R}$

Dimostrare che ha punti di max/min assoluto in $\mathbb{R}$.

Abbiamo visto che $\lim_{x\to\infty} f(x) = 0$

e che per $x \geq 0$ $f(x) > 0$ e che per su $x < -1$ si ha $f(x) < 0$

$$f(x) = \frac{[x]+1}{[x]^2 + [x] + 1}$$

Allora esistono $x_0 < -1$ e $x_1 > 0$

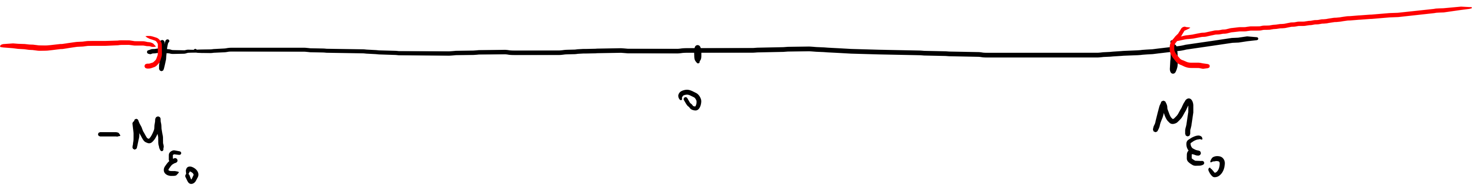
t.c. $f(x_0) < 0 < f(x_1)$

$$0 < \varepsilon_0 = \frac{1}{2} \min\{|f(x_0)|, |f(x_1)|\}$$

$$\lim_{x \to \infty} f(x) = 0 \Longrightarrow \boxed{\forall \varepsilon > 0 \ \exists M_\varepsilon > 0 \text{ t.c. } |x| > M_\varepsilon \Longrightarrow |f(x)| < \varepsilon}$$

Consideriamo ora $\varepsilon_0 = \frac{1}{2}\min\{|f(x_0)|, |f(x_1)|\}$.

Allora $|x| > M_{\varepsilon_0}$ ho $|f(x)| < \varepsilon_0$



Per $|x| > M_0 \Longrightarrow -\varepsilon_0 < f(x) < \varepsilon_0$

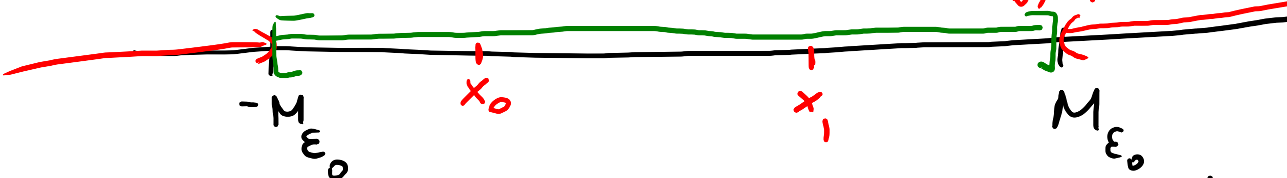
Siccome $f(x_0) = -|f(x_0)| < -\frac{1}{2}|f(x_0)| \le -\varepsilon_0 < f(x)$

Per $|x| > M_{\varepsilon_0}$ $\quad f(x) < \varepsilon_0 = \frac{1}{2} \min\{|f(x_0)|, |f(x_1)|\}$

$$\le \frac{1}{2} |f(x_1)| = \frac{1}{2} f(x_1) < f(x_1)$$

Conclusione: $\boxed{|x| > M_{\varepsilon_0} \Longrightarrow f(x_0) < f(x) < f(x_1).}$

$\Downarrow$ $x_0, x_1 \in [-M_{\varepsilon_0}, M_{\varepsilon_0}]$



Consideriamo $f|_{[-M_{\varepsilon_0}, M_{\varepsilon_0}]}$. La funzione

$[x]$ assume solo un numero finito di valori in $[-M_{\varepsilon_0}, M_{\varepsilon_0}]$

$[-M_{\varepsilon_0}], [-M_{\varepsilon_0}]+1, \dots, [M_{\varepsilon_0}]$

$\Longrightarrow$ $f(x) = \dfrac{[x]+1}{[x]^4+[x]+1}$ assume un numero finito di valori in $[-M_{\varepsilon_0}, M_{\varepsilon_0}]$

$= f([-M_{\varepsilon_0}, M_{\varepsilon_0}]) = \{ f(x) : x \in [-M_{\varepsilon_0}, M_{\varepsilon_0}] \}$ ha un numero finito di elementi $\Rightarrow \exists \max f([-M_{\varepsilon_0}, M_{\varepsilon_0}])$

ed $\exists \min f([-M_{\varepsilon_0}, M_{\varepsilon_0}])$

cioè $\exists$ x_m e x_M t.c.

$$\boxed{f(x_m) \le f(x) \le f(x_M) \quad \forall \; x \in [-M_{\varepsilon_0}, M_{\varepsilon_0}]}$$

$$f(x_m) \le f(x_0) < 0 < f(x_1) \le f(x_M)$$

Per concludere che x_m è il punto di minimo assoluto su tutto $\mathbb{R}$, bisogna dimostrare che

$$f(x_m) \leq f(x) \quad \forall\ x \in \mathbb{R}$$

mentre, al momento, sappiamo che

$$\boxed{f(x_m) \leq f(x) \quad \forall\ x \in [-M_{\varepsilon_0}, M_{\varepsilon_0}]}$$

Per $x \notin [-M_{\varepsilon_0}, M_{\varepsilon_0}]$ osservo che

$$f(x_m) \leq f(x_0) < f(x)$$

Similmente, x_M si dimostra è un punto di massimo

assoluto in $\mathbb{R}$

<u>Lemma</u> Sia $p(x)$ polinomio di grado $\leq n$ e sia $x_0 \in \mathbb{R}$. Allora, se $p(x) = o\left((x-x_0)^n\right)$ in x_0, si ha $p \equiv 0$

Corollario Sia f t.c. $f(x_0), \dots, f^{(n)}(x_0)$ esistono in un punto x_0. Sia $p(x)$ un polinomio di grado $\leq n$ t.c. $f(x) = p(x) + o((x-x_0)^n)$.

Allora $$p(x) = \underbrace{\sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j}_{P_n(x)}$$

Dim Da Peano

$$f(x) = P_n(x) + o((x-x_0)^n).$$ Se ora

$$f(x) = p(x) + o((x-x_0)^n)$$ allora

$$P(x) + o((x-x_0)^n) = P_n(x) + o((x-x_0)^n)$$

$$P(x) - P_n(x) = o((x-x_0)^n)$$

$P(x) - P_n(x)$ e' un polinomio di grado $\leq n$.

$$\overset{\text{Lemma}}{\Longrightarrow} \quad P(x) = P_n(x) \quad \forall x.$$

__Esempio__ Calcoliamo tutti i polinomi di Mc Laurin

di $f(x) = x^2 \sin(x^3)$.

$$\sin(y) = \sum_{j=0}^{n} \frac{(-1)^j \, y^{2j+1}}{(2j+1)!} + o\left(y^{2n+1}\right)$$

$$y = x^3$$

$$x^2 \sin(x^3) = \left(\sum_{j=0}^{n} \frac{(-1)^j \, x^{6j+3}}{(2j+1)!} + o\left(x^{6n+3}\right) \right) x^2$$

$$x^2\sin(x^3) = \underbrace{\sum_{j=0}^{n} \frac{(-1)^j \, x^{6j+5}}{(2j+1)!}}_{P_{6n+5}(x)} + o\left(x^{6n+5}\right)$$

$n=0$ $\quad P_5(x) = x^5$

$$P_0 = P_1 = P_2 = P_3 = P_4 = 0$$

$$x^2\sin(x^3) = x^5 + o(x^5)$$

$$x^2\sin(x^3) = o(x^4)$$

$$x^2\sin(x^3) = o(x^3)$$

$$f(x) = x^2 \sin(x^3) = \underbrace{\sum_{j=0}^{n} (-1)^j \frac{x^{6j+5}}{(2j+1)!}}_{P_{6n+5}(x)} + o(x^{6n+5})$$

$$x^2 \sin(x^3) = \boxed{\sum_{j=0}^{n+1} (-1)^j \frac{x^{6j+5}}{(2j+1)!}} + o(x^{6n+11}) \qquad P_{6n+11}$$

$\underline{P_{6n+6}}$, P_{6n+7}, P_{6n+8}, P_{6n+9}, $P_{6n+10} = P_{6n+5}$

$f^{(k)}(0) = ?$ $\qquad P_{6n+5}(x) = \sum_{k=0}^{6n+5} \frac{f^{(k)}(0)}{k!} x^k$

$$\boxed{P_{6n+5}(x) = \sum_{j=0}^{n} (-1)^j \frac{x^{6j+5}}{(2j+1)!} = \sum_{k=0}^{6n+5} \frac{f^{(k)}(0)}{k!} x^k}$$

Se k non è della forma $\cancel{k = 6j+5}$ allora $f^{(k)}(0) = 0$

Se invece $k = 6j+5$

$$\frac{f^{(6j+5)}(0)}{(6j+5)!} = \frac{(-1)^j}{(2j+1)!} \qquad f^{(6j+5)}(0) = \frac{(-1)^j (6j+5)!}{(2j+1)!}$$

$f(x) = e^x \sin x \qquad P_4(x)$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) = x - \frac{x^3}{3!} + o(x^4)$$

$$f(x) = \left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) + o(x^4)\right)\left(\left(x - \frac{x^3}{3!}\right) + o(x^4)\right) =$$

$$= o(x^4) + x - \frac{x^3}{3!} + x^2 - \cancel{\frac{x^4}{3!}} + \frac{x^3}{2} + \cancel{\frac{x^4}{3!}} = x + x^2 + \underbrace{\left(\frac{1}{2} - \frac{1}{6}\right)}_{\frac{1}{3}} x^3 + o(x^4)$$

$\lg(\cos x)$ $\qquad P_4$

$$\lg(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + o(y^3) = \sum_{j=1}^{n} (-1)^{j-1}\frac{y^j}{j} + o(y^n)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$$

$$\lg(\cos x) = \lg\left(1\left(-\frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)\right)\right) =$$

$$= -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{\left(-\frac{x^2}{2} + \cancel{\frac{x^4}{4!}}\right)^2}{2} + \cancel{\frac{\left(-\frac{x^2}{2} + \frac{x^4}{4!}\right)^3}{3}} - \cancel{\frac{\left(-\frac{x^2}{2} + \frac{x^4}{4!}\right)^4}{4}} + o(x^4)$$

$$= -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{1}{2}\,\frac{x^4}{4} + o(x^4)$$