Stimatori campionari di

valore atteso

varianza

covarianza

correlazione

Data X variabile aleatoria con

$$E(X) = \mu_X \qquad var(X) = \sigma_X^2$$

si raccolgono N campioni $x(1), x(2), x(3) \dots x(N)$

Valore atteso → media

come stimare il valore atteso?

$$\boxed{\bar{X} = \frac{1}{N} \sum_{i=1}^{N} x(i)}$$

stimatore non polarizzato e consistente

Varianza

definizione $\sigma_X^2 = E\{[X - E(X)]^2\}$

$= E\{X^2\} - [E(X)]^2$

stimatore non polarizzato

$$\boxed{S_X^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left(x(i) - \bar{X}\right)^2}$$

media aritmetica

Correlazione

Date 2 v.a. X, Y si definisce in generale

$$\gamma_{X,Y}(t_1, t_2) = E\left[x(t_1) \cdot y(t_2)\right]$$

e nel caso di processi stazionari

$$\gamma_{XY}(\tau) = E\left[x(t) \cdot y(t+\tau)\right]$$

τ qualsiasi

Correlazione campionaria

Supponiamo di aver raccolto N campioni dei processi di X ed Y

Covarianza

Date 2 v.a. X, Y con

$E(X) = \mu_x$

$E(Y) = \mu_y$

$var(X) = \sigma_x^2$

$var\,Y = \sigma_y^2$

Si definisce

$$\sigma_{xy} = E\left[(X-\mu_x)(Y-\mu_y)\right]$$

Supponendo di avere N campioni sia di X che di Y

$$\boxed{S_{xy} = \frac{1}{N-1}\sum_{i=1}^{N} \left(x(i)-\overline{X}\right)\left(y(i)-\overline{Y}\right)}$$

N campioni per X ed Y

$x(1) \quad x(2) \quad x(3) \quad \cdots \quad x(N)$

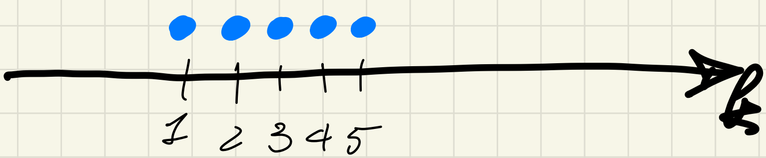
$y(1) \quad y(2) \quad y(3) \quad \cdots \quad y(N)$

$$\boxed{\hat{\gamma}(k) = \frac{1}{N-|k|} \sum_{i=1}^{N-|k|} x(i)\, y(i+k)}$$

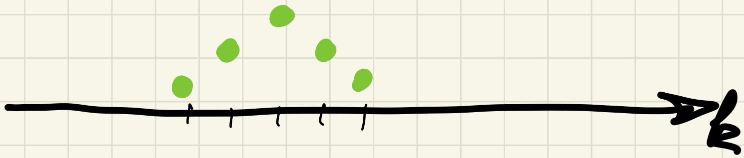
stimatore non polarizzato

Esempio $\Rightarrow$ 5 campioni

$x(i)$

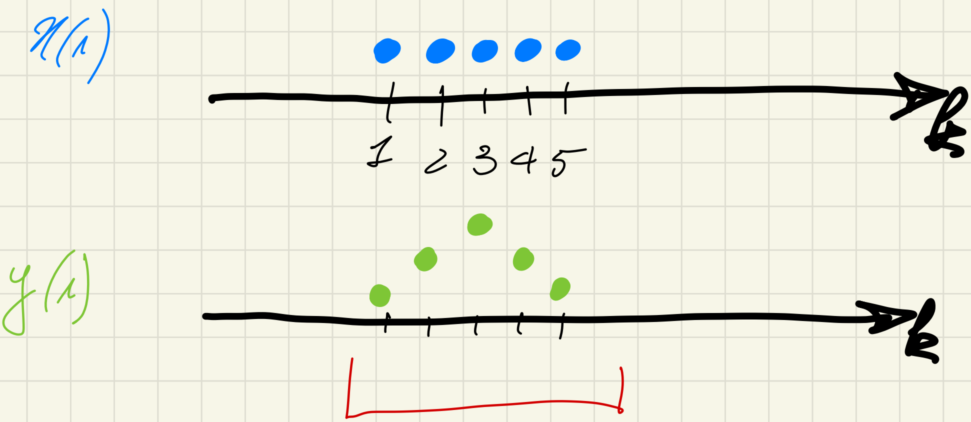


$y(i)$



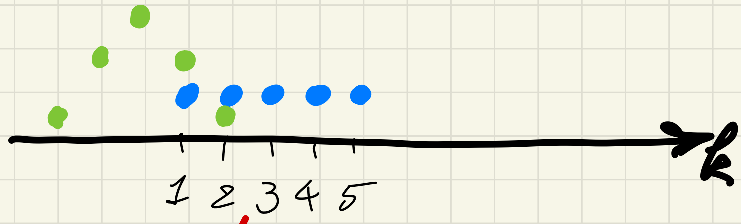
$$\hat{r}(k) = \frac{1}{N-|k|} \sum_{i=1}^{N-|k|} x(i)\,y(i+k)$$

per $k=0$ $\quad \hat{r}(0) = \sum_{i} x(i)\,y(i)$



5 contributi $x(i)\,y(i)$

$k=3$



2 contributi

$x(i)\,y(i+k)$

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8 campioni $y(t)$

k	0	1	2	3	4	5	6	7
y	3,211	2,316	3,447	4,257	3,447	0,722	0,568	-0,712

$\hat{\gamma}(0)$ $\quad\hat{\gamma}(1)$ $\quad\hat{\gamma}(2)$ $\qquad E[y(t)]=0$

$$\hat{\gamma}(0) = \frac{1}{N-|k|}\sum_{i=0}^{N-|k|-1} y(i)\,y(i+k) = \frac{1}{8}\sum_{i=0}^{7}[y(i)]^2 = 7,357$$

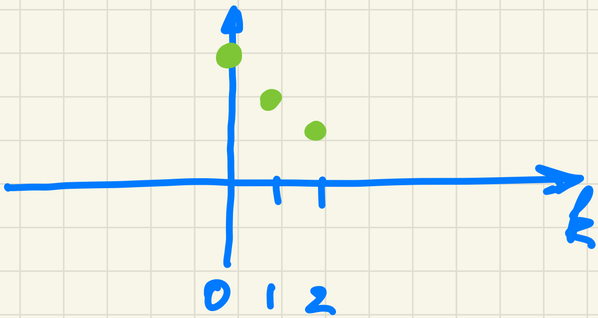
$k=0$

$$\hat{\gamma}(1) = \frac{1}{N-|k|} \sum_{i=0}^{N-|k|-1} y(i)\,y(i+k) =$$

$k=1$

$$= \frac{1}{7} \sum_{i=0}^{6} y(i)\,y(i+1) = 6{,}746$$

$$\hat{\gamma}(2) = \frac{1}{6} \sum_{i=0}^{5} y(i)\,y(i+2) = 6{,}218$$



$\to$ MA(1) ?

$\to$ AR(1)

se fosse MA(1) $\gamma(\pm 2) = 0$

$\gamma(\pm\tau) = 0$

$|\tau| > 1$

M $\quad y(t) = a\,y(t-1) + \eta(t) \qquad \eta(\cdot) \sim WN(0, \lambda^2)$

$\hat{a} = ? \quad \hat{\lambda}^2 = ?$

equazione Yule Walker

$\hat{\gamma}(\tau) \xrightarrow[N \to \infty]{} \gamma(\tau)$

$\gamma(1) = a\gamma(0)$

$\hat{\gamma}(1) = \hat{a}\hat{\gamma}(0) \Rightarrow \hat{a} = \dfrac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \simeq 0,92$

$\gamma(2) = a\gamma(1)$

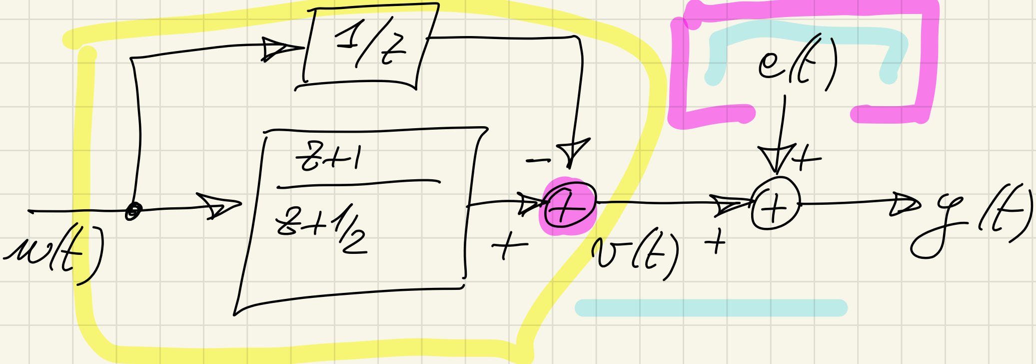
$\hat{a} = \dfrac{\hat{\gamma}(2)}{\hat{\gamma}(1)} \simeq 0,92$

$\hat{\lambda}^2 = ?$

$$\gamma(0) = a\gamma(-1) + \lambda^2$$
$$= a\gamma(1) + \lambda^2$$

$$\hat{\gamma}(0) = \hat{a}\hat{\gamma}(1) + \hat{\lambda}^2$$

$$\hat{\lambda}^2 = \hat{\gamma}(0) - \hat{a}\hat{\gamma}(1) \simeq 1,1507$$



$u(\cdot) \sim WN(0,1)$

$e(\cdot) \sim WN(0,1)$

→ NON correlati

(a) FdT tra u ed y

(b) spettro $y(\cdot)$

$$H_{u,y}(z) = \left(\frac{z+1}{z+\frac{1}{2}} - \frac{1}{z}\right) = \frac{z^2 - \frac{1}{2}}{z\left(z+\frac{1}{2}\right)} \quad (a)$$

$$\boxed{\underbrace{H(z)\, u}_{v}} + F(z) \cdot e = y$$

$$\boxed{y(t) = v(t) + e(t)}$$

$$H(z) = \frac{z^2 - 1/2}{\boxed{z(z+1/2)}}$$

$u(t) \longrightarrow \boxed{H(z)} \longrightarrow v(t)$

$v(\cdot)$ è stazionario?

$E(v) = ?$

$z(z+1/2)$

poli $\left|\frac{z}{2}\right| < 1$

il sistema $H(z)$ è as stabile

A regime $v(t)$ è processo stoc. stazionario

è in forma canonica? Sì

$y(t) = e(t) + v(t)$

$y(\cdot)$ è stazionario?

$E[y(t)] = E[e(t)] + E[v(t)]$

$= 0 + 0 = 0$

$y(t_1) = e(t_1) + v(t_1)$

$y(t_2) = e(t_2) + v(t_2)$

$\longrightarrow E[y(t_1) \cdot y(t_2)] = \longrightarrow ?$

$v(t) = H(z)u(t)$

$v(t) = \sum_{i=0}^{+\infty} h(i)u(t-i)$

$$E\left[y(t_1)\cdot y(t_2)\right] = E\left\{\left[e(t_1) + \underbrace{\sum_{i=0}^{+\infty} h(i)\,u(t_1-i)}_{v(t_1)}\right]\cdot\left[e(t_2) + \underbrace{\sum_{i=0}^{+\infty} h(i)\,u(t_2-i)}_{v(t_2)}\right]\right\}$$

$$= E\left[e(t_1)\cdot e(t_2)\right] + E\left[v(t_1)\cdot v(t_2)\right]$$

$$+ \boxed{\sum_{i=0}^{+\infty} h(i)\cdot\left\{E\left[v(t_1)\cdot u(t_2-i)\right]\right\}} +$$

$$\cancel{+ \sum_{i=0}^{+\infty} h(i)\cdot\left\{\boxed{E\left[e(t_1)\cdot u(t_2-i)\right]}\right\}} + \;0$$

$$\cancel{+ \sum_{i=0}^{+\infty} h(i)\cdot\left\{\boxed{E\left[e(t_2)\cdot u(t_1-i)\right]}\right\}} \;0$$

$e(\cdot)$ $u(\cdot)$ scorrelati

$$E\left[y(t_1)\cdot y(t_2)\right] = \boxed{\underbrace{E\left[e(t_1)\cdot e(t_2)\right]}_{\text{stazionario}}} + \boxed{\underbrace{E\left[v(t_1)\cdot v(t_2)\right]}_{\text{stazionario}}}$$

$$\gamma_y(t_1, t_2) = \gamma_e(t_2 - t_1) + \gamma_v(t_2 - t_1)$$

$$\Downarrow$$

$$\boxed{\gamma_y(t_2-t_1) = \gamma_e(t_2-t_1) + \gamma_v(t_2-t_1)}$$

$y(\,)$ stazionario

$$\Gamma_y(\omega) = \Gamma_e(\omega) + \Gamma_v(\omega)$$

$\Gamma_e(\omega) = 1 \quad \leftarrow$ rumore bianco $WN(0,1)$

$$\Gamma_v(\omega) = \left|H(e^{j\omega})\right|^2 \cdot \Gamma_u(\omega) = \left|H(e^{j\omega})\right|^2 \cdot 1$$

$u(\cdot) \sim WN(0,1)$

$$\Gamma_y(\omega) = 1 + \left|H(e^{j\omega})\right|^2$$

$$\left|H(e^{j\omega})\right|^2 = \frac{\left|e^{2j\omega} - \frac{1}{2}\right|^2}{\left|e^{j\omega}\right|^2 \cdot \left|e^{j\omega} + \frac{1}{2}\right|^2} = \frac{\frac{5}{4} - \cos(2\omega)}{\frac{5}{4} + \cos(\omega)}$$

$$\Gamma_y(\omega) = 1 + \quad = \frac{5/2 + \cos\omega - \cos 2\omega}{\frac{5}{4} + \cos\omega}$$

processo staz stazionario $\quad E[v(t)] = 0$

$v(\cdot)$

$$\begin{cases} \gamma(0) = 5 \\ \gamma(\pm 1) = 2 \\ \gamma(\pm \tau) = 0 \quad \tau = 2, 3, 4 \ldots \end{cases}$$

Determinare per il processo una descrizione come processo MA di ordine opportuno.

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ordine del processo MA ?

$$MA(1) \Longleftarrow \begin{cases} \gamma(0) \neq 0 \\ \gamma(\pm 1) \neq 0 \\ \gamma(\pm \tau) \equiv 0 \quad |\tau| \geq 2 \end{cases}$$

$$v(t) = c_0\, e(t) + c_1\, e(t-1) \qquad e(\cdot) \sim WN(0, \lambda^2)$$

$$= e(t) + c\, e(t-1)$$

$$\gamma(0) = var[v(t)] = (c_0^2 + c_1^2)\lambda^2 = \boxed{(1 + c^2)\lambda^2 = 5}$$

$$\gamma(1) = \boxed{2 = c\lambda^2}$$

$$\begin{cases} (1+c^2)\,d^2 = +5 \\ c\,d^2 = +2 \end{cases}$$

$$\boxed{d^2 = \frac{2}{c}}$$

$$(1+c^2)\,\frac{2}{c} = 5$$

$$2c^2 - 5c + 2 = 0$$

$$c_{1,2} = \frac{5 \pm 3}{4} \quad \nearrow 2 \quad \searrow 1/2$$

(a) $c = +2$, $d^2 = +1$

$$e(\cdot) \to \boxed{C(z)} \to v(\cdot)$$

$$C_1(z) = 1 + c z^{-1} = \frac{z+c}{z}$$

zero $\to$ $-c \to -2$ $\quad |\cdot| > 1$

(b) $e(\cdot) \to \boxed{C_2(z)} \to v(\cdot)$

$$C_2(z) = \frac{z+c}{z}$$

$c = 1/2$