Solutions

November 16, 2020

Exercise 1. Determine the image set of each function, then draw its graph and determine its inverse function.

$$f(x) = \begin{cases} -(x-1)^2 & \text{if } x < 1 \\ \sqrt{x-1} & \text{if } x \geq 1 \end{cases} \qquad g(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ -\sqrt{x} - 1 & \text{if } x \geq 0 \end{cases}$$

$$h(x) = \begin{cases} x^3 - 2 & \text{if } x < 0 \\ \sqrt{x} + 2 & \text{if } x \geq 0 \end{cases} \qquad i(x) = \begin{cases} \dfrac{x}{x+1} & \text{if } x < -1 \\ 1 - (x+1)^2 & \text{if } x \geq -1 \end{cases}$$

Solutions

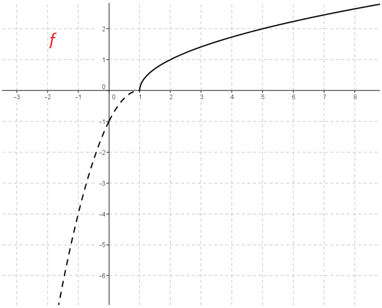
$$f^{-1} : \mathbb{R} \to \mathbb{R} \qquad f^{-1}(y) = \begin{cases} 1 - \sqrt{-y} & \text{if } y < 0 \\ y^2 + 1 & \text{if } y \geq 0 \end{cases}$$

$$g^{-1} : \mathbb{R} \to \mathbb{R} \qquad g^{-1}(y) = \begin{cases} (1+y)^2 & \text{if } y \leq -1 \\ -\sqrt{1+y} & \text{if } y > -1 \end{cases}$$

$$h^{-1} :]-\infty, -2[\cup [2, +\infty[\to \mathbb{R} \qquad h^{-1}(y) = \begin{cases} \sqrt[3]{y+2} & \text{if } y < -2 \\ (y-2)^2 & \text{if } y \geq 2 \end{cases}$$

$$i^{-1} : \mathbb{R} \to \mathbb{R} \qquad i^{-1}(y) = \begin{cases} \sqrt{1-y} - 1 & \text{if } y \leq 1 \\ \frac{y}{1-y} & \text{if } y > 1 \end{cases}$$

The graphs of the functions f, g, h, i are represented in the next page.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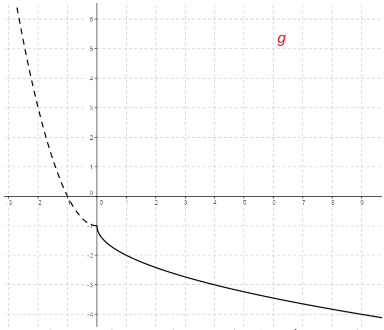
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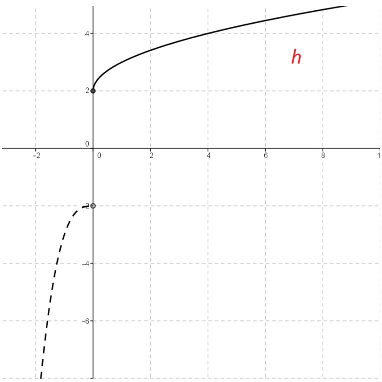
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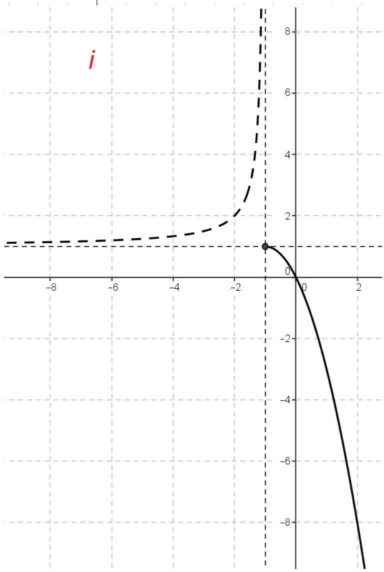

i

Exercise 2. Determine the domain of the following function:

$$f(x) = \arcsin\left(\frac{1}{\sqrt{x+5}-\sqrt{x}}\right).$$

Solution
The domain of f is $[0,4]$.

Exercise 3. Consider the following function, depending on the parameters a and b:

$$f:\mathbb{R}\to\mathbb{R}, \quad f(x) = \begin{cases} -x+4 & \text{if } x<-2, \\ -2\sin(ax)+b & \text{if } -2\le x\le 0, \\ x^a-2x & \text{if } 0<x. \end{cases}$$

a) For which values of a and b the conditions $f(-1)=2$ and $f(3)=-5$ are satisfied?

b) Draw the graph of the function obtained.

c) Is the function injective? Is it surjective? If not, determine a possible modification of f in the interval $[-2,0]$ so that to obtain a monotone function.

Solutions

a) $a=0, b=2$.

b) The graph of the function f obtained with the choices $a=0$ and $b=2$ is the following.

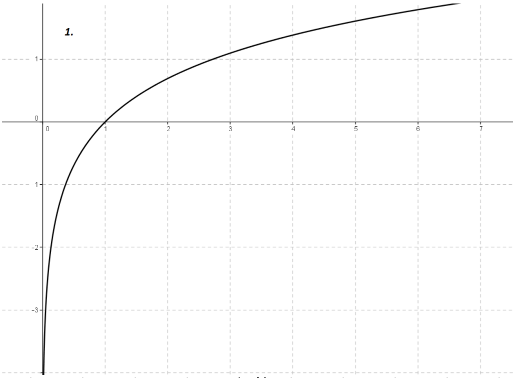
c) The function f is not injective, since in the interval $[-2,0]$ the value of f is constantly 2; the function is not surjective anymore, since

$$f(\mathbb{R}) =]-\infty,1[\,\cup\{2\}\cup\,]6,+\infty[.$$

A possible modification of f in the interval $[-2,0]$ in order to make the function bijective is

$$\bar{f}(x) = 1-\tfrac{5}{2}x, \quad \text{for all } x\in[-2,0].$$

Exercise 4. Recognise which graphs represent monotone functions and, for each of the remaining ones, determine the maximal intervals of monotonicity.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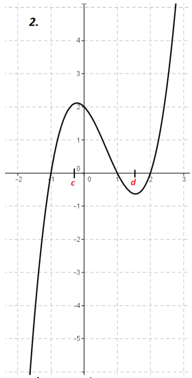
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c
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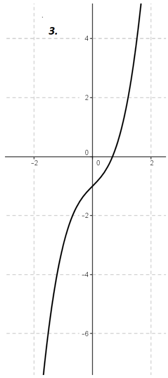
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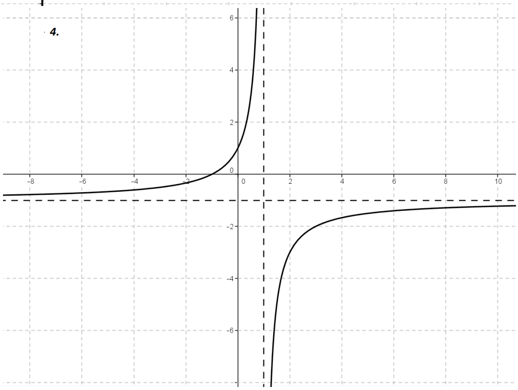
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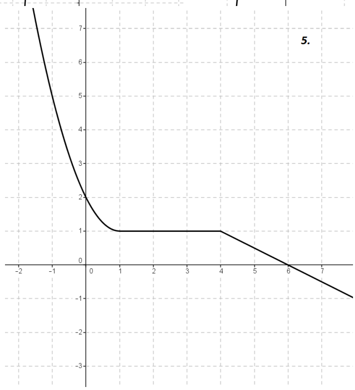
5.

Solutions

1. The function is increasing in its domain, which is $]0, +\infty[$, then it is strictly monotone in $]0, +\infty[$.
2. The function is increasing in $]-\infty, c]$ and in $[d, +\infty[$, decreasing in $[c, d]$ (see the graph: c is a local maximum point, d a local minimum point of the function).
3. The function is increasing in $\mathbb{R}$, then it is strictly monotone in $\mathbb{R}$.
4. The function is increasing in $]-\infty, 1[$ and in $]1, +\infty[$, but not in the whole domain $]-\infty, 1[\, \cup \,]1, +\infty[$.
5. The function is non-increasing in $\mathbb{R}$, then it is monotone in $\mathbb{R}$, in particular it is decreasing in $]-\infty, 1]$ and in $[4, +\infty[$, constant (non-decreasing and non-increasing) in $[1, 4]$.