30 Novembre

$\lg(1+x)$ ne calcoleremo più avanti i polinomi di McLaurin

$\operatorname{artg} x$, stesso cosa.

Altri esempi di polinomi di McLaurin

Es1 $f(x) = \dfrac{1}{1-x}$. E' simile a $\dfrac{1}{1+x} = (1+x)^{-1}$

Noi sappiamo già che

$$(1+x)^{-1} = \sum_{j=0}^{n} \binom{-1}{j} x^j + o(x^n)$$

$$(1-x)^{-1} = \sum_{j=0}^{n} \binom{-1}{j} (-1)^j x^j + o(x^n) \quad *$$

(dove ho usato $o((-x)^n) = o((-1)^n x^n) = o(x^n)$),

Possiamo scrivere $*$ in modo più esplicito

Ricordiamo la formula

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$$\boxed{\frac{1}{1-x} = \sum_{j=0}^{n} x^j + \frac{x^{n+1}}{1-x}}$$

dove $\lim_{x\to 0} \dfrac{\frac{x^{n+1}}{1-x}}{x^n} = \lim_{x\to 0} \dfrac{x^{n+1}}{x^n(1-x)} =$

$= \lim_{x\to 0} \dfrac{x}{1-x} = \dfrac{0}{1} = 0$, cioè $\dfrac{x^{n+1}}{1-x} = o(x^n)$

$$\frac{1}{1-x} = \sum_{j=0}^{n} x^j + o(x^n), \quad o(x^n) = \frac{x^{n+1}}{1-x}.$$

$\Longrightarrow$ è il polinomio di McLaurin di ordine n di $\frac{1}{1-x}$.

$$\frac{1}{1+x} = \sum_{j=0}^{n} (-1)^j x^j + o(x^n)$$

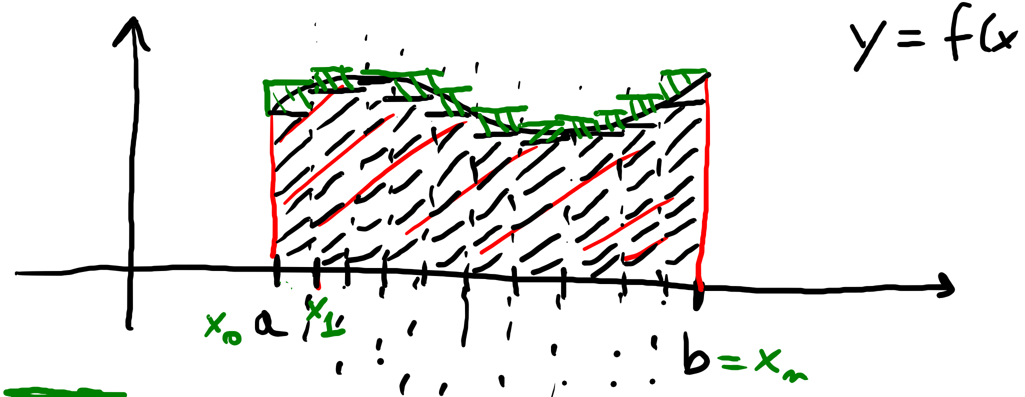
è il pol. di McLaurin di ordine n di $\frac{1}{1+x}$

$$\frac{1}{1+x} = \sum_{j=0}^{n} (-1)^j x^j + o(x^n)$$

$$\frac{1}{1+x^2} = \underbrace{\sum_{j=0}^{n} (-1)^j x^{2j}}_{P_{2n}(x)} + o(x^{2n})$$

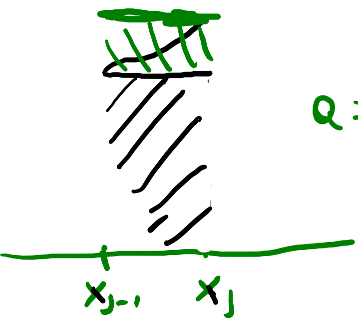
$P_{2n}(x)$ polinomi di McLaurin di $\dfrac{1}{1+x^2}$

L'integrale (di Darboux)



$$\int_a^b f(x)\,dx$$

$$\left(\int f(x)\,dx\right)$$



$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

Integrale di Darboux.

Riguardo funzioni $f : [a,b] \to \mathbb{R}$, dove $a < b$ sono numeri reali, che sono limitate, cioè $\inf f([a,b]) > -\infty$, e $\sup f([a,b]) < +\infty$

(Equivalentemente, 1) esistono m ed M in $\mathbb{R}$ t.c.

$$m \leq f(x) \leq M \quad \forall \quad x \in [a,b]$$

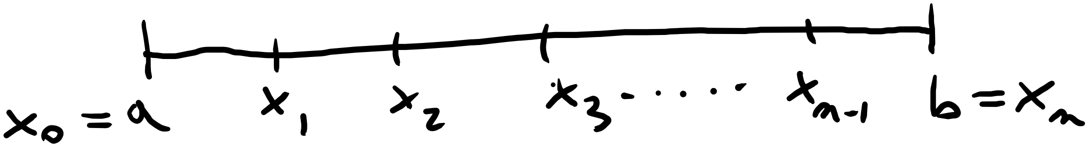
2) esiste $M \in \mathbb{R}$ t.c. $|f(x)| \leq M \quad \forall$
$x \in [a,b]$)

Def Una decomposizione Δ di $[a,b]$ e' della forma

$$\Delta = \{[x_0,x_1],\dots,[x_{n-1},x_n]\}$$

dove

$$x_0=a<x_1<\dots<x_{n-1}<x_n=b$$



Il calibro di Δ e'

$$|\Delta| = \max\{x_1-x_0,\dots,x_n-x_{n-1}\}$$

Def Date due decomposizioni di $[a,b]$

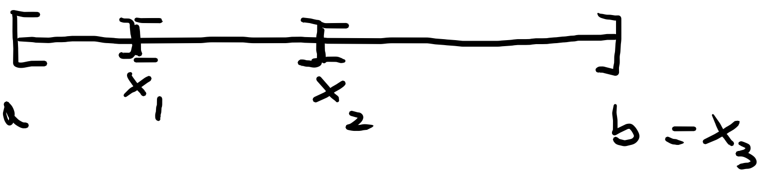
Δ $\qquad x_0 = a < x_1 < \dots < x_{n-1} < x_n = b$

Δ' $\qquad y_0 = a < y_1 < \dots < y_{n'-1} < y_{n'} = b$

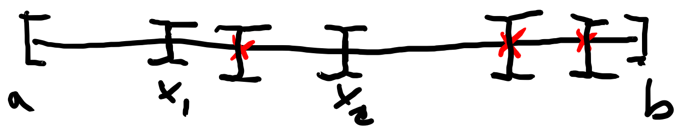
diciamo che Δ' è un raffinamento di Δ se

se $\{x_0, x_1, \dots, x_n\} \subseteq \{y_0, \dots, y_{n'}\}$

Esempio



Δ

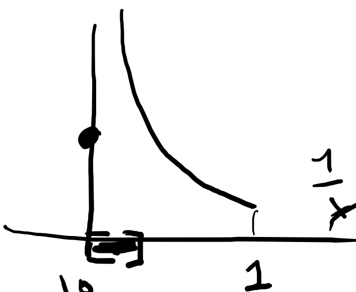


Δ'

$\Delta' \preceq \Delta$

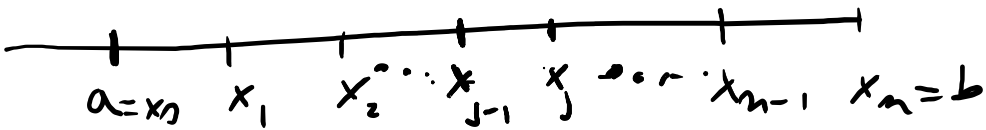
Quando Δ' è un raffinamento di Δ scriverò $\Delta' \preceq \Delta$

Def Sia $f : [a,b] \to \mathbb{R}$ limitata,

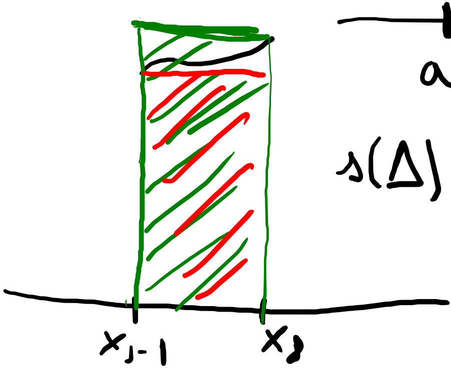


$$\Delta : \quad a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup f([x_{j-1}, x_j])$$



$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf f([x_{j-1}, x_j])$$



Se $m \le f(x) \le M \quad \forall x \in [a,b]$ e se

Δ : $x_0 = a < \dots < x_n = b$,

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \underbrace{\inf f([x_{j-1}, x_j])}_{\ge m} \ge \sum_{j=1}^{n} (x_j - x_{j-1})\, m$$

$$= (b-a)\, m$$

$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \underbrace{\sup f([x_{j-1}, x_j])}_{\le M} \le \sum_{j=1}^{n} (x_j - x_{j-1})\, M$$

$$= (b-a)\, M$$

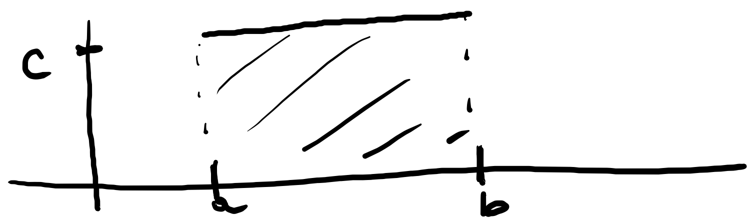
$$\boxed{(b-a)\, m \le s(\Delta) \le S(\Delta) \le M(b-a) \quad \forall \Delta}$$

Esempio $f \equiv c$ in $[a,b]$

$$c \leq f(x) \leq c \quad \forall\, x \in [a,b]$$

$$(b-a)c \leq s(\Delta) \leq S(\Delta) \leq (b-a)c \qquad \forall\, \Delta.$$

$$\Rightarrow \quad s(\Delta) = S(\Delta) = (b-a)c\,.$$



Esempio $\quad D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ $\quad [a,b]$

$\Delta: \quad x_0 = a < \cdots < x_n = b$ $\quad \boxed{S(\Delta) = b-a}$

$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup D([x_{j-1}, x_j]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup\{0,1\}$$

$$= \sum_{j=1}^{n} (x_j - x_{j-1}) = b-a$$

$$D([x_{j-1}, x_j]) \subseteq D(\mathbb{R}) = \{0,1\}$$

$$D([x_{j-1}, x_j]) \ni 1, 0 \Rightarrow D([x_{j-1}, x_j]) = \{0,1\}$$

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf D([x_{j-1}, x_j]) =$$

$$= \sum_{j=1}^{n} (x_j - x_{j-1}) \underbrace{\inf \{0, 1\}}_{0} = 0$$

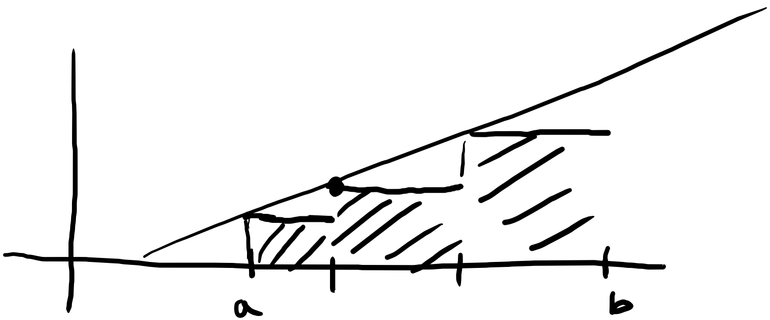
$D(x|$

$$s(\Delta) = 0 \qquad S(\Delta) = b - a$$

Sia $f:[a,b]\to\mathbb{R}$ crescente $\Rightarrow f(a)\le f(x)\le f(b)$ $\forall\, x\in[a,b]$

Δ : $x_0=a<\dots\dots<x_n=b$

$$s(\Delta)=\sum_{j=1}^{n}(x_j-x_{j-1})\,\underbrace{\inf f([x_{j-1},x_j])}_{f(x_{j-1})}=\sum_{j=1}^{n}(x_j-x_{j-1})\,f(x_{j-1})$$



$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \underbrace{\sup f([x_{j-1}, x_j])}_{f(x_j)} =$$

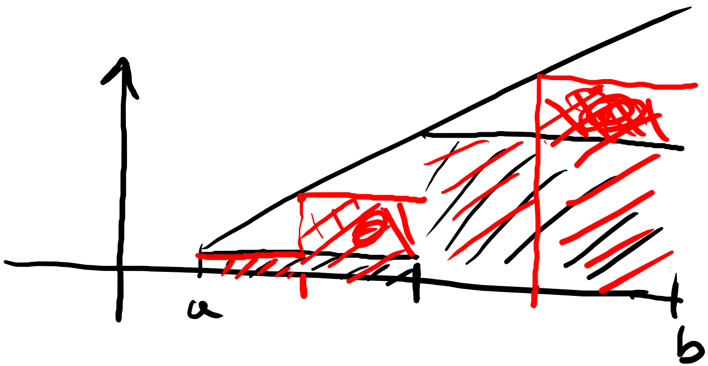
$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) f(x_j)$$

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) f(x_{j-1})$$

$B = \{ S(\Delta) \}_\Delta$ $\qquad$ $A = \{ s(\Delta) \}_\Delta$ sono una coppia separata

__Lemma__ Data $f:[a,b]\to\mathbb{R}$ limitata e date due decomposizioni $\Delta' \succeq \Delta$. Allora

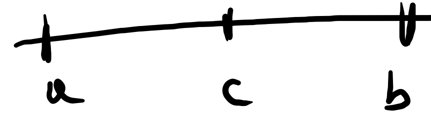
$$s(\Delta)\le s(\Delta')\le S(\Delta')\le S(\Delta)$$



Dim Dimostriamo solo $S(\Delta') \leq S(\Delta)$

per

$$\Delta = \{[a,b]\}$$



$$\Delta' = \{[a,c],[c,b]\}$$

$$S(\Delta) = (b-a)\sup f([a,b])$$

$$S(\Delta') = (c-a)\sup f([a,c]) + (b-c)\sup f([c,b])$$

Siccome $f([a,c]) \subseteq f([a,b])$, $f([c,b]) \subseteq f([a,b])$

allora $\sup f([a,c]) \leq \sup f([a,b])$

$\sup f([c,b]) \leq \sup f([a,b])$

$$S(\Delta') = (c-a)\sup f([a,c]) + (b-c)\sup f([c,b])$$

$$\leq (c-a)\sup f([a,b]) + (b-c)\sup f([a,b])$$

$$= (b-a)\sup f([a,b]) = S(\Delta)$$