30 Novembre

Preso $\Omega \subseteq \mathbb{R}^d$ introduciamo $C^{k,\alpha}(\Omega)$

$\alpha \in (0,1)$, $k = 0, 1, \dots$

è il sottospazio di $W^{k,\infty}(\Omega) \cap C^k(\Omega)$

t.c.

$$\sup_{\substack{x \neq y \\ \text{in } \Omega}} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x-y|^\alpha} < +\infty$$

$\forall \ |\mu| = k$

B_R

Lemma Sia $u \in L^\infty((0,T), L^2(B_R))$. Allora $\forall R' < R$:

1) Se per $\beta \in [2,\infty]$ ho $w \in L^\beta((0,T), W^{k,\infty}(B_R)) \Longrightarrow$
$u \in L^\beta((0,T), W^{k,\infty}(B_{R'}))$

2) Se per $\alpha \in (0,1)$ e $k \geq 0$ ho $w \in L^\beta((0,T), C^{k,\alpha}(B_R)) \Longrightarrow$
$u \in L^\beta((0,T), C^{k+1,\alpha'}(B_{R'})) \quad \forall \alpha' \in (0,\alpha)$.

Dim
$k=0$ Fisso $R'' \in (R', R)$. Vale

$$u(x) = \boxed{-\frac{1}{4\pi}\int_{B_{R''}} \frac{x-y}{|x-y|^3} \times w(y)\, dy} + h(x)$$ $h(x)$ armonico in $B_{R''}$

$$h \in L^{\beta}((0,T), W^{m,\infty}(B_{R'})) \qquad \forall\, m \in \mathbb{N}.$$

$x \in B_R$

$$\left| \int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y)\, dy \right| \leq \int_{B_{R''}} \frac{1}{|x-y|^2}\, dy \quad |\omega|_{L^\infty_x(B_R)}$$

$$\leq \underbrace{\int_{B_{2R''}(x)} \frac{1}{|x-y|^2}\, dy}_{\leq C_{R''}} \qquad |\omega|_{L^\infty(B_R)}$$

$|u|$ $\qquad u = \tilde{u} + h$

$$|\tilde{u}|_{L^\beta(0,T)L^\infty(B_{R'})} \lesssim |\omega|_{L^\beta(0,T)L^\infty(B_R)}$$

$$\omega \in L^3\big((0,T), C^{k,\alpha}(B_R)\big) \to u \in L^3\big((0,T), C^{k+1,\alpha'}(B_{R'})\big)$$

$k=0$

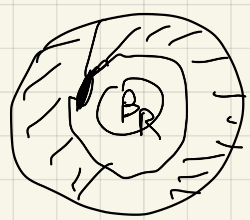
$$\tilde{u}(x) = \frac{1}{4\pi}\int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y)\,dy$$

$$\varphi \in C_c^\infty(B_{R''}, [0,1]) \qquad \varphi|_{B_{R'}} = 1$$

$$\tilde{u}(x) = \boxed{\frac{1}{4\pi}\int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y)\,\varphi(y)\,dy} + \frac{1}{4\pi}\int_{B_{R'}} \frac{x-y}{|x-y|^3} \times \omega(y)\,(1-\varphi(y))\,dy$$

supp ha distanza positiva di B_R

$$\boxed{v(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)\,dy}$$



Lemma Sia $K \in C^\infty(\mathbb{R}^d \setminus \{0\})$ omogenea di grado $-(d-1)$.

Allora

$$\langle \partial_j K, \varphi \rangle = P.V. \int_{\mathbb{R}^d} \partial_j K(y)\, \varphi(y)\, dy - c_j\, \varphi(0)$$

$$P.V. \int_{\mathbb{R}^d} \partial_j K(y)\, \varphi(y)\, dy = \lim_{\varepsilon \to 0^+} \int_{|y| \geq \varepsilon} \partial_j K(y)\, \varphi(y)\, dy$$

$$c_j = \int_{|x|=1} K(x)\, x_j\, dS$$

Dim

$$\langle \partial_j K, \varphi \rangle = -\langle K, \partial_j \varphi \rangle = -\int_{\mathbb{R}^d} K(y)\, \partial_j \varphi(y)\, dy =$$

$$= -\lim_{\varepsilon \to 0^+} \int_{|y| \geq \varepsilon} K(y)\, \partial_j \varphi(y)\, dy =$$

$$= \cancel{-} \lim_{\varepsilon \to 0^+} \left[\cancel{-} \int_{|y| \geq \varepsilon} \partial_j K(y)\, \varphi(y)\, dy + \int_{|y|=\varepsilon} K(y)\, \varphi(y)\, \frac{\cancel{-}y_j}{|y|}\, dS \right]$$

$$\lim_{\varepsilon\to 0^+} \int_{|y|=\varepsilon} \psi(y) K(y) \frac{y_j}{|y|} \, dS \qquad y = \varepsilon z \qquad |z|=1$$

$$= \lim_{\varepsilon\to 0^+} \int_{|z|=1} \psi(\varepsilon z) K(\cancel{\varepsilon} z) \, z_j \, \cancel{\varepsilon^{d-1}} \, dS = \psi(0) \int_{|z|=1} K(z) z_j \, dS$$

$$\boxed{\langle \partial_j K, \psi \rangle = \boxed{\lim_{\varepsilon\to 0^+} \int_{|y|\geq\varepsilon} \partial_j K(y) \, \psi(y) \, dy} + c_j \, \psi(0)}$$

$$\int_{|y|=1} \partial_j K(y) \, dS = 0$$

$$\partial_j K(y) \sim \frac{1}{|y|^d}$$

$$\int_{|y|\geq\varepsilon} \underline{\partial_j K(y) \, \psi(y) \, dy}$$

$w \in C_c^\infty(B_R)$

$$v(x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times w(y)\, dy = \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times w(x-y)\, dy$$

$$\nabla_x v(x) = \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \nabla_x w(x-y)\, dy =$$

$$= - \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \nabla_y w(x-y)\, dy$$

$$= - \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \nabla_y w(y)\, dy =$$

$$= P.V. \int_{\mathbb{R}^3} \nabla_y \frac{x-y}{|x-y|^3} \times w(y)\, dy + L w(x)$$

$$|Lw|_{C^{0,\alpha}(B_{R'})} \leq |Lw|_{C^{0,\alpha}(\mathbb{R}^3)} \leq |L|\ |w|_{C^{0,\alpha}(\mathbb{R}^3)}$$

$$= |L|\ |w|_{C^{0,\alpha}(B_R)}$$

$$\text{P.V.} \int_{\mathbb{R}^3} \left(\nabla_y \frac{x-y}{|x-y|^3} \right) \times w(y)\, dy \qquad w \in C^{0,\alpha}$$

$$\Rightarrow u \in C^{1,\alpha'}$$

$$\int_{\mathbb{R}^3} H(x-y) \times w(y)\, dy \qquad u \in W^{1,\infty}(B_{R'})$$

$$\int_{|y|=1} H(y)\, dS = 0$$

$$x \in B_R$$

$$|\nabla v(x)| = PV \int H(x-y) \times w(y)\, dy = \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} H(x-y) \times w(y)\, dy$$

$$= \lim_{\varepsilon \to 0^+} \left| \int_{\varepsilon \leq |y| \leq 2R} (H(y)) \times (w(x-y) - w(x))\, dy \right|$$

$$\int_{\varepsilon \leq |y| \leq 2R} \underbrace{|H(y)|}_{\frac{1}{|y|^3}} \, |w|_{C^{0,\alpha}} \, |y|^\alpha \, dy \xrightarrow{\varepsilon \to 0} |w|_{C^{0,\alpha}(B_R)} \int_{|y| \leq 2R} |H(y)|\, |y|^\alpha dy$$

$|x| \leq R'$
$$|\nabla v(x)| \leq C \left([w]_{C^{0,\alpha}(B_R)} + |w|_{L^\infty(B_R)} \right)$$

$$\nabla v \in C^{0,\alpha'}(B_{R'})$$

$x, x' \in B_{R'}$
$$|\nabla v(x) - \nabla v(x')| =$$

$$= \left| P.V. \int_{\mathbb{R}^3} H(x-y)\, w(y)\, dy - P.V. \int_{\mathbb{R}^3} H(x'-y)\, w(y)\, dy \right|$$

$$= \left| P.V. \int_{B_{2R}} H(y) \left(w(x-y) - w(x'-y) \right) dy \right| =$$

$$= \cancel{P.V.} \int_{B_{2R}} |H(y)| \, \left| w(x-y) - w(x) + w(x') - w(x'-y) \right| dy$$

$$|w(x-y) - w(x) + w(x') - w(x'-y)| \leq 2 [w]_{C^{0,\alpha}(B_R)} \min\{|y|^\alpha, |x'-x|^\alpha\}$$

Se $|y| < |x - x'|$

$$|w(x-y) - w(x) + w(x') - w(x'-y)| \leq$$

$$|w(x-y) - w(x)| + |w(x') - w(x'-y)| \leq 2 |y| [w]_{C^{0,\alpha}(B_R)}$$

Se invece $|y| > |x - x'|$

$$\leq |w(x-y) - w(x'-y)| + |w(x) - w(x')| \leq 2 |x-x'| [w]_{C^{0,\alpha}}$$

$$|\nabla v(x) - \nabla v(x')| \lesssim |w|_{C^{0,\alpha}(B_R)} \int_{B_{2R}} \frac{1}{|y|^3} \min\{|y|^\alpha, |x'-x|^\alpha\} \, dy$$

$$\lesssim \int_{|y| \le |x-x'|} |y|^{\alpha-3} \, dy + \int_{|x'-x| \le |y| \le 2R} \frac{1}{|y|^3} \, dy \; |x'-x|^\alpha$$

$$\sim \int_0^{|x-x'|} r^{\alpha-1} \, dr + \int_{|x'-x|}^{2R} r^{-1} \, dr \; |x'-x|^\alpha$$

$$\sim |x-x'|^\alpha + |x'-x|^\alpha \log \frac{2R}{|x'-x|}$$

$$\Rightarrow \quad \nabla v \in C^{0,\alpha'}(B_{R'}) \qquad 0 < \alpha' < \alpha$$

$$w \in \underbrace{C_c^\infty}(B_R) \qquad \underline{C_c^{0,\alpha}(B_R)}$$

Lemma $\forall$ $u \in L^r(\mathbb{R}^3, \mathbb{R}^3)$ $r \in (2, \infty)$

$\exists !$ $p \in L^{\frac{r}{2}}(\mathbb{R}^3)$ t.c.

$$-\Delta p = \partial_i \partial_j (u_i u_j)$$

p è dato $p = \frac{\partial_i}{\sqrt{-\Delta}} \frac{\partial_j}{\sqrt{-\Delta}} (u_i u_j) = R_i R_j (u_i u_j)$

R_i gli operatori di Reisz.

$$|p|_{L^{\frac{r}{2}}} \leq \sum_{ij} |R_i R_j (u_i u_j)|_{L^{\frac{r}{2}}} \lesssim$$

$$\lesssim \sum_{ij} |u_i u_j|_{L^{\frac{r}{2}}} \lesssim |u|^2_{L^r} .$$

Prop Sia u una soluzione debole di NS con $u \in L^\infty(\mathbb{R}_+, L^2)$ $\nabla u \in L^2(\mathbb{R}_+, L^2)$, allora, posto $p = R_i R_j (u_i u_j)$, si ha

$p \in L^1(\mathbb{R}_+, L^3)$ (segue da $u \in L^2(\mathbb{R}_+, L^6)$)

ed inoltre u è una soluzione distribuzionale di

$$\boxed{\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p}$$

$$\langle R_i R_j \partial_k (u^k u^j), \phi_i \rangle = \langle \partial_i \underbrace{R_j R_k (u^k u^j)}_{P}, \phi_i \rangle$$

$$\langle \nabla P, \phi \rangle$$

$$\frac{\partial_i}{\sqrt{-\Delta}} \frac{\partial_j}{\sqrt{-\Delta}} \partial_k = \partial_i \frac{\partial_j}{\sqrt{-\Delta}} \frac{\partial_k}{\sqrt{-\Delta}}$$

Local Serrin

Teo Considero u soluzione debole di Leray-Hopf e supponiamo che in un aperto $U \subseteq \mathbb{R}^3$ ho

$$u \in L^r((0,T), L^s(U)) \qquad \frac{2}{r} + \frac{3}{s} \leq 1$$

con $r \geq 2$ e $s \geq 3$ e dove escludo $(r,s) = (0,3)$.

Allora $u \in C^\infty_x([0,T] \times U, \mathbb{R}^3)$. Più precisamente

$\forall \ \Omega \subset \overline{\Omega} \subset\subset U$ e $\forall \ t_0 \in (0,T]$

$u \in L^\infty((t_0,T), H^k(\Omega))$, $\forall \ k \in \mathbb{N}$,

$u \in C^{0,\gamma}_t([t_0,T], C^0_x(\Omega))$ $\forall \ \gamma \in (0, \frac{1}{2})$

$u \in C^{0,\gamma_k}_t([t_0,T], C^k_x(\Omega))$ γ_k