MATHEMATICS CLASS

from November 12 to December 3, 2020

Exercise 1. For each of the following periodic functions, find its minimum period.

a. $g(x) = \frac{1}{2}\sin\left(\frac{x}{3}\right)$ *b.* $i(x) = \tan(\pi x)$ *c.* $h(x) = \sin(-2x)$

Solutions

a. 6π *b.* 1 *c.* π

Exercise 2. Let us consider the following function:

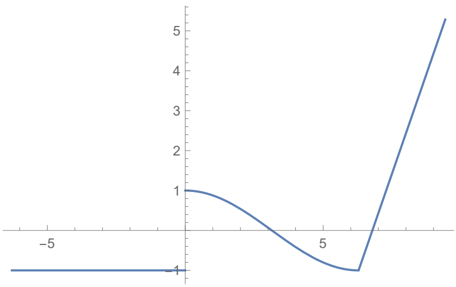
$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ \cos\left(\frac{x}{2}\right) & \text{if } 0 \le x < 2\pi, \\ 2x - (4\pi + 1) & \text{if } x \ge 2\pi. \end{cases}$$

i) Draw the graph of the function f.

ii) Is the function continuous at $x = 0$ and $x = 2\pi$? Justify your answers.

iii) Determine a possible modification of the function f in a neighbourhood of $x = 0$ or of $x = 2\pi$, so that to obtain a continuous function in the whole $\mathbb{R}$.

Solutions



i)

ii) f is not continuous at $x = 0$, whereas it is continuous at $x = 2\pi$.

iii) A possible modification of f in a neighbourhood of $x = 0$ is given by $\bar{f} : \mathbb{R} \to \mathbb{R}$, with

$$\bar{f}(x) = \begin{cases} -1 & \text{if } x < -1, \\ 2x + 1 & \text{if } -1 \le x < 0, \\ \cos\left(\frac{x}{2}\right) & \text{if } 0 \le x < 2\pi, \\ 2x - (4\pi + 1) & \text{if } x \ge 2\pi. \end{cases}$$

Exercise 3. Let us consider the following functions, defined in $\mathbb{R} \setminus \{0\}$:

$$f(x) = 1, \qquad g(x) = \begin{cases} 1 & \text{if } x < 0, \\ 2 & \text{if } x > 0. \end{cases}$$

Is it possible to extend them so that to make them continuous at $x = 0$? If so, how? If not, why?

Solutions

f is continuous in $\mathbb{R} \setminus \{0\}$ and the unique way to extend f in order to obtain a continuous function in $\mathbb{R}$ is defining $\bar{f}(0) = 1$, and $\bar{f}(x) = f(x)$, for all $x \in \mathbb{R} \setminus \{0\}$;
g is continuous in $\mathbb{R} \setminus \{0\}$ but it is not extendible to a continuous function in $\mathbb{R}$.

Exercise 4. Let us consider $f : [0, 4] \to f([0, 4])$, which is a continuous, strictly monotone function, such that $f(0) = -2$ and $f(4) = 3$.

i) According to the sign of f at the extreme points of the domain, which type of monotonicity in $[0, 4]$ does f display?

ii) How many times does the graph of f intersect the x-axis? How many the graph of the inverse f^{-1}? Justify your answers, by means of some theoretical results studied.

Solutions

i) f is increasing.

ii) By the Bolzano theorem and the strictly monotonicity follows the graph of f intersects once time the x-axis. The inverse f^{-1} is continuous by the Theorem on the continuity of the inverse, therefore the same conclusion as f holds.

Exercise 5. Let us consider the following function:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{x+2} & \text{if } x < -2, \\ (x+2)(x-4) & \text{if } -2 \le x < 4, \\ \frac{1}{2}(4 - x) & \text{if } 4 \le x. \end{cases}$$

Determine

i) the sign of f;

ii) the image of f, $f(\mathbb{R})$;

iii) $\lim_{x \to -\infty} f(x), \quad f(-3), \quad \lim_{x \to (-2)^-} f(x), \quad f(-2), \quad \lim_{x \to 4} f(x), \quad f(6), \lim_{x \to +\infty} f(x).$

Moreover say whether the function f is continuous in its domain and draw its graph.

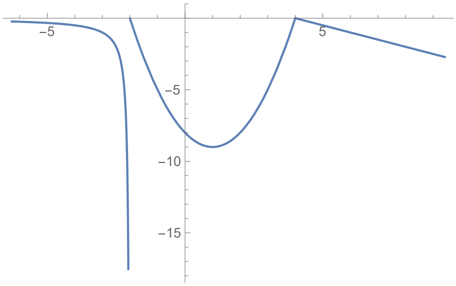
Solutions

i) $f(x) \le 0$ for all $x \in \mathbb{R}$, with $f(x) = 0$ if and only if $x \in \{-2, 4\}$;

ii) $f(\mathbb{R}) =]-\infty, 0]$;

iii) $\lim_{x \to -\infty} f(x) = 0, \quad f(-3) = -1, \quad \lim_{x \to (-2)^-} f(x) = -\infty, \quad f(-2) = 0,$
$\lim_{x \to 4} f(x) = 0, \quad f(6) = -1, \quad \lim_{x \to +\infty} f(x) = -\infty.$

The function f is continuous in $]-\infty, -2[\cup]-2, +\infty[$, but not at $x = -2$, since $\lim_{x \to (-2)^-} f(x) = -\infty$. The graph of f is represented below:



Exercise 6. Compute the following limits, whenever it is possible:

1) $\lim_{x\to -1}(x^3-3)$, 2) $\lim_{x\to 3}\frac{x^2-5x+6}{x-3}$, 3) $\lim_{x\to\pi}\cos(3x)$,

4) $\lim_{x\to+\infty} x\cos\left(\frac{1}{x}\right)$, 5) $\lim_{x\to 0}\frac{7}{x}$, 6) $\lim_{x\to 0}\frac{7}{x^2}$,

7) $\lim_{x\to+\infty}\frac{\arctan x}{3}$, 8) $\lim_{x\to 0}\sqrt[3]{1+\frac{1}{x}}$, 9) $\lim_{x\to+\infty}\frac{\cos^2 x}{x}$.

Solutions

1) -4, 2) 1, 3) -1,

4) $+\infty$, 5) it does not exist, 6) $+\infty$,

7) $\frac{\pi}{6}$, 8) it does not exist, 9) 0.

Exercise 7. Compute the following limits of sequences, whenever it is possible:

1) $\lim_{n\to+\infty}\frac{4-n^2}{n-2}$, 2) $\lim_{n\to+\infty}\frac{9n^2+2}{6-n+n^2}$, 3) $\lim_{n\to+\infty}\frac{n-2-3n^2}{4-5n^2+6n^3}$,

4) $\lim_{n\to+\infty}\left(\sqrt{n-5}-\sqrt{n+3}\right)$, 5) $\lim_{n\to+\infty}(-1)^n\frac{\arctan n}{n}$, 6) $\lim_{n\to+\infty}\frac{(-1)^n(n+\pi)}{3\pi-n+\sqrt{3n}}$,

7) $\lim_{n\to+\infty}(-1)^n\frac{7\sqrt[4]{n}}{n^2+5+\tan(\frac{1}{n})}$, 8) $\lim_{n\to+\infty}\frac{n-\sqrt{n+1}}{n+\sqrt{n+1}}$, 9) $\lim_{n\to+\infty}\frac{n+\arccos(\frac{1}{n})}{n^3}$,

10) $\lim_{n\to+\infty}\frac{n}{\sin(2n\pi)+\cos((2n+1)^2\pi)}$.

Solutions

1) $-\infty$, 2) 9, 3) 0,

4) 0, 5) 0, 6) it does not exist,

7) 0, 8) 1, 9) 0,

10 $-\infty$.

Exercise 8. The following limits are indeterminate forms. Compute them by applying suitable relevant limits, when they are useful.

1) $\lim_{x\to 0} \frac{\sin(2x^2+7x)}{x}$, 2) $\lim_{x\to -\infty} \frac{x^2+\sin x}{x^2+2x-5}$, 3) $\lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{x(x^4+2)}$,

4) $\lim_{x\to 0} \sin\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)\frac{x\sin x}{\cos x - 1}$, 5) $\lim_{x\to -\infty} \frac{x^5+2x^3+1}{x^2+7x+4}$, 6) $\lim_{x\to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x}$,

7) $\lim_{x\to -\infty} \frac{(1-x^2)(1+x^2)}{x-x^4}$, 8) $\lim_{x\to +\infty} \left(\frac{x^2}{x+1} - x\right)$, 9) $\lim_{x\to -1} \frac{x^2-2x-3}{\arcsin(x+1)}$,

10) $\lim_{x\to 0} \frac{\arctan(5x) - 1 + \cos x}{x(x+\pi)}$, 11) $\lim_{x\to \frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x}$.

Solutions

1) 7, 2) 1, 3) $\frac{1}{4}$,

4) $-\sqrt{2}$, 5) $-\infty$, 6) $\frac{\sqrt{2}}{2}$,

7) 1, 8) -1, 9) -4,

10) $\frac{5}{\pi}$, 11) $-\sqrt{2}$.

Exercise 9. Compute the following limits.

1) $\lim_{x\to +\infty} \frac{2x}{x+e^{-x}}$, 2) $\lim_{x\to +\infty} \frac{e^{\frac{1}{x}}-1}{\sin\frac{2}{x}}$, 3) $\lim_{x\to 0} \frac{e^x - e}{x^2-1}$,

4) $\lim_{x\to 0} \frac{e^{x^2}-1}{1-\sqrt{x^2+1}}$, 5) $\lim_{x\to 0} \frac{1-\sqrt{e^x+1}}{\log(1+x)}$, 6) $\lim_{x\to 0^+} \frac{e^{x^2-x}-1}{x}$,

7) $\lim_{x\to 0^+} \frac{e^x-1}{\sqrt{1-\cos x}}$, 8) $\lim_{x\to +\infty} \frac{\log(x^2-x)}{x^2-1}$, 9) $\lim_{x\to 0} \frac{\log(x^2+1)}{1-\cos x}$,

10) $\lim_{x\to +\infty} \frac{\log(x^2)}{\log(x)}$, 11) $\lim_{x\to +\infty} \frac{x^2-\log(x^3-3)}{e^x - \sin x + x^2}$, 12) $\lim_{x\to +\infty} \frac{e^x - \log x}{e^{x^2} - x^{33}}$,

13) $\lim_{x\to 0} \frac{e^x-1}{e^{4x}-1}$, 14) $\lim_{x\to +\infty} \left(1+\frac{2}{x+1}\right)^x$, 15) $\lim_{x\to +\infty} \left(\frac{x+3}{x+1}\right)^x$.

Solutions

1) 2, 2) $\frac{1}{2}$, 3) $e-1$,

4) -2, 5) it does not exist, 6) -1,

7) $\sqrt{2}$, 8) 0, 9) 2,

10) 2, 11) 0, 12) 0,

13) $\frac{1}{4}$, 14) e^2, 15) e^2.