

Preimage of a set through a function

$$X, Y \subset \mathbb{R} \quad f: X \rightarrow Y$$

$$B \subset Y \quad f^{-1}(B) = \{x \in X : f(x) \in B\} \quad \text{preimage of } B \\ \text{through } f$$

(contrainimmagine di B
tramite la funzione f)

Exercise 2.4.4 (Chapter 2, from the book)

$$h(x) = x^2 - 4x \quad B = [5, +\infty[\quad \text{dom}(h) = \mathbb{R}$$

$$h^{-1}(B) = ? \quad h^{-1}(B) = \{x \in \text{dom}(h) : h(x) \in B\} = \{x \in \mathbb{R} : x^2 - 4x \geq 5\}$$

$$x^2 - 4x - 5 \geq 0 \iff x \leq -1 \vee x \geq 5$$

$$\Delta = 16 + 20 = 36 = 6^2$$

$$x_1 = \frac{4-6}{2} = -1 \quad x_2 = \frac{4+6}{2} = 5$$

$$h^{-1}(B) = \{x \in \mathbb{R} : x \leq -1 \vee x \geq 5\} =]-\infty, -1] \cup [5, +\infty[$$

$$i(x) = \sqrt{3x-1}$$

$$B = [0, 1[$$

$$i^{-1}(B) = ?$$

$$i^{-1}(B) = \{x \in \text{dom}(i) : i(x) \in B\} = \left\{x \in \left[\frac{1}{3}, +\infty[: \sqrt{3x-1} \in [0, 1[\right\}$$

$$\text{dom}(i) = \{x \in \mathbb{R} : 3x-1 \geq 0\} = \left[\frac{1}{3}, +\infty[$$

$$\begin{cases} \sqrt{3x-1} \geq 0 \\ \sqrt{3x-1} < 1 \end{cases} \text{ is obvious because } \sqrt{\cdot} : [0, +\infty[\rightarrow [0, +\infty[$$

$$\sqrt{3x-1} < 1 \iff (\sqrt{3x-1})^2 < 1^2$$

$$\iff 3x-1 < 1$$

$$\iff x < \frac{2}{3}$$

$$\frac{1}{6-x} < 0$$

↕

$$6-x < 0 \iff x > 6$$

$$j^{-1}(B) = \left\{ x \in \left[\frac{1}{3}, +\infty[: x < \frac{2}{3} \right\} = \left[\frac{1}{3}, \frac{2}{3}[$$

$$j(x) = \frac{1}{6-x} \quad B =]-\infty, 0[$$

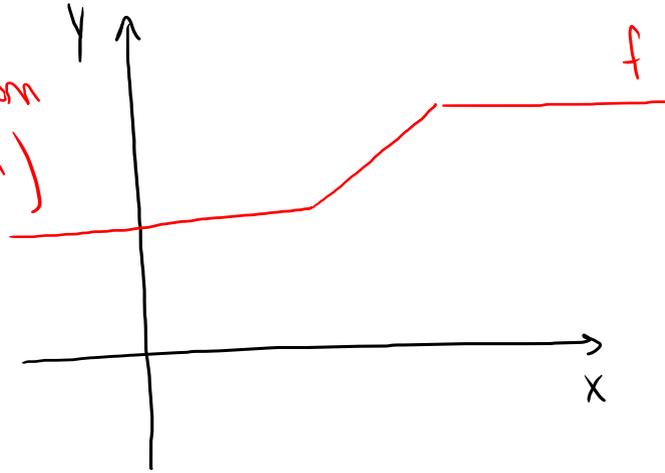
$$\text{dom}(j) = \mathbb{R} \setminus \{6\}$$

$$j^{-1}(B) = ? \quad j^{-1}(B) = \left\{ x \in \text{dom}(j) : j(x) \in B \right\} = \left\{ x \in \mathbb{R} \setminus \{6\} : \frac{1}{6-x} < 0 \right\}$$

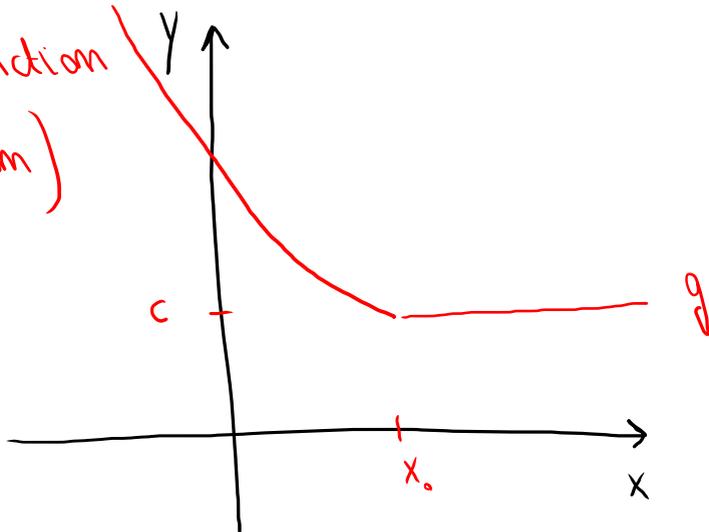
$$=]6, +\infty[$$

Examples of monotone functions

1) Weakly increasing function
(or nondecreasing function)



2) Weakly decreasing function
(or nonincreasing function)



3) Increasing functions

$$f(x) = ax + b \quad a > 0 \quad \text{on } \mathbb{R}$$
$$b \in \mathbb{R}$$

$$g(x) = x^3 \quad \text{on } \mathbb{R}$$

$$h(x) = \sqrt{x} \quad \text{on } [0, +\infty[$$

4) Decreasing functions

$$j(x) = ax + b \quad a < 0 \quad \text{on } \mathbb{R}$$
$$b \in \mathbb{R}$$

$$k(x) = -x^3 \quad \text{on } \mathbb{R}$$

$$l(x) = -\sqrt{x} \quad \text{on } [0, +\infty[$$

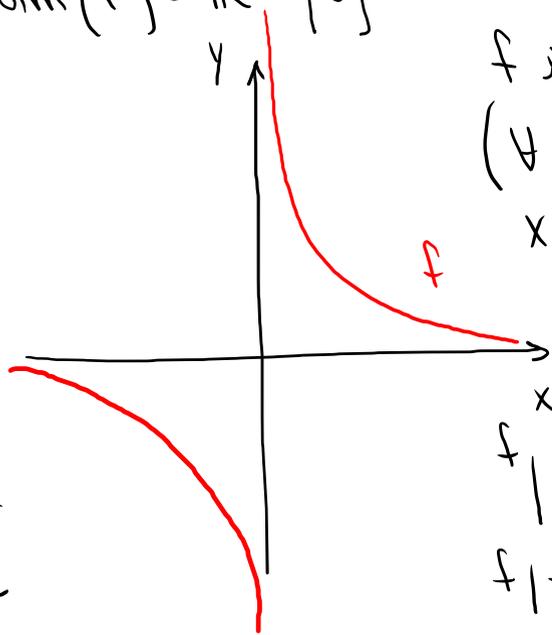
Property: if f is weakly (strictly) increasing on $A \subset \mathbb{R}$ then $-f$ is weakly (strictly) decreasing on A .

Property: If $f: A \rightarrow B$ is strictly monotone
(increasing or decreasing), then f is injective.

$f: A \rightarrow B$ injective ~~\implies~~ f is strictly monotone.

Counterexample

$$f(x) = \frac{1}{x} \quad \text{dom}(f) = \mathbb{R} \setminus \{0\}$$



f is injective
 $(\forall x_1, x_2 \in \text{dom}(f),$
 $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$

BUT
it is not true

that f is
decreasing

in $]-\infty, 0[\cup]0, +\infty[$

$f|_{]-\infty, 0[}$ is decreasing
 $f|_{]0, +\infty[}$ is decreasing

Exercise 4.4.3 (from Chapter 4, of the book)

Find if any set is lower / upper bounded;

for any set, find its infimum and its supremum; its minimum and its maximum (if they exist).

$$A = \{2, 3, 5, 7, 11\}$$

A is lower bounded: $\forall x \in A, 2 \leq x$ and $2 \in A$
 $\rightarrow \underline{2 = \min A = \inf A}$

A is upper bounded: $\forall x \in A, x \leq 11$ and $11 \in A$
 $\rightarrow \underline{11 = \max A = \sup A}$

$$B = [2, \pi]$$

B is lower bounded: $\forall x \in B, x \geq 2$ and $2 \in B$

$$\longrightarrow \underline{2 = \min B = \inf B}$$

B is upper bounded: $\forall x \in B, x \leq \pi$ and $\pi \in B$

$$\longrightarrow \underline{\pi = \max B = \sup B}$$

$$C =]0, e[$$

C is lower bounded: $\forall x \in C, x \geq 0$ (that is, 0 is a lower bound of C)

0 is largest lower bound of C

$$\begin{array}{l} 0 = \underline{\inf C} \\ 0 \notin C \quad \underline{\nexists \min C} \end{array}$$

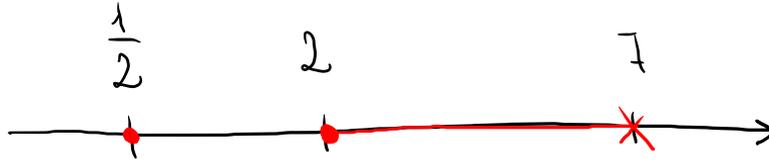
C is upper bounded: $\forall x \in C, x \leq e$ (e is an upper bound of C)
 e is the smallest upper bound of C

$$\underline{e = \sup C}$$

$$e \notin C$$

$$\nexists \max C$$

$$E = \left\{ \frac{1}{2} \right\} \cup [2, 7[$$



E is lower bounded: $m \leq \frac{1}{2}$ is a lower bound of E

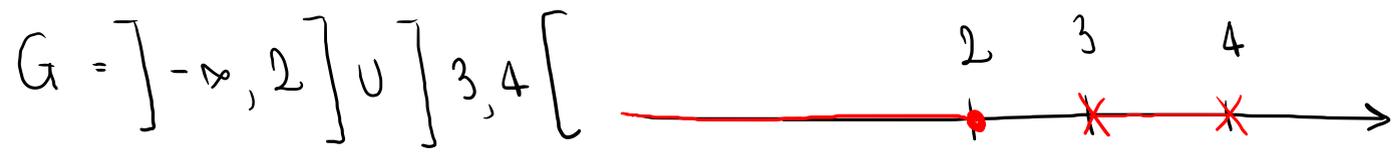
$\frac{1}{2}$ is the largest lower bound of E

$$\left. \begin{array}{l} \underline{\frac{1}{2} = \inf E} \\ \frac{1}{2} \in E \end{array} \right\} \Rightarrow \underline{\frac{1}{2} = \min E}$$

E is upper bounded: $M \geq \forall$ is an upper bound of E

\exists is smallest upper bound of E

$$\left. \begin{array}{l} \underline{\exists = \sup E} \\ \exists \notin E \end{array} \right\} \Rightarrow \underline{\nexists \max E}$$



G is lower unbounded: $\forall m \in \mathbb{R}, \exists x \in G$ such that $x < m$ (there doesn't exist any real lower bound of G)

$\rightarrow \underline{\inf G = -\infty}$

G is upper bounded: $M \geq 4$ is an upper bound of G

4 is the smallest upper bound of G

$$\left. \begin{array}{l} 4 = \sup G \\ 4 \notin G \end{array} \right\} \Rightarrow \nexists \max G$$

Exercise 2 (November 16, 2020)

Find the domain of

$$f(x) = \arcsin \left(\frac{1}{\sqrt{x+5} - \sqrt{x}} \right)$$

$\sin: \mathbb{R} \rightarrow [-1, 1]$ is periodic

$\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}: [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ is increasing and surjective

then it is invertible and

\arcsin is its inverse function.

$$\arcsin: [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



dom (f)

$$-1 \leq \frac{1}{\sqrt{x+5} - \sqrt{x}} \leq 1$$

argument of arcsin

~~$$x+5 \geq 0$$~~

$$x \geq 0$$

$$\sqrt{x+5} - \sqrt{x} \neq 0$$

• Notice that:

$$x + 5 \geq 0 \iff x \geq -5 \text{ is implied by } x \geq 0$$

$$x \geq 0$$

• Notice also that:

$$\sqrt{x+5} - \sqrt{x} > 0$$

is always satisfied for all $x \geq 0$

$$\sqrt{x+5} > \sqrt{x}$$

$$(\sqrt{x+5})^2 > (\sqrt{x})^2$$

$$x+5 > x$$

$$5 > 0 \text{ (true)}$$

$$\begin{cases} \frac{1}{\sqrt{x+5} - \sqrt{x}} \leq 1 \\ x \geq 0 \end{cases} \iff \begin{cases} \sqrt{x+5} - \sqrt{x} \geq 1 \\ x \geq 0 \end{cases}$$

$$\begin{aligned} &\sqrt{x+5} - \sqrt{x} \geq 1 \\ \iff &\sqrt{x+5} \geq 1 + \sqrt{x} \end{aligned}$$

$$\iff (\sqrt{x+5})^2 \geq (1 + \sqrt{x})^2$$

$$\cancel{x+5} \geq 1 + 2\sqrt{x} + \cancel{x}$$

$$4 \geq 2\sqrt{x} \iff 2 \geq \sqrt{x} \iff 4 \geq x$$

$$0 \leq x \leq 4$$

$$\text{dom}(f) = [0, 4]$$