$$\int_\gamma \langle g, \tau \rangle \, ds = \int_a^b \langle g(\gamma(t)), \gamma'(t) \rangle \, dt$$

<u>OSS</u>: Siano γ_1 e γ_2 curve equivalenti $\quad \gamma_2(t) = \gamma_1(h(t)) \quad \boxed{h : [a_2, b_2] \to [a_1, b_1]}$

$$\int_{\gamma_2} \langle g, \tau \rangle \, ds = \int_{a_2}^{b_2} \langle g(\gamma_2(t)), \gamma_2'(t) \rangle \, dt = \int_{a_2}^{b_2} \langle g(\gamma_1(h(t))), \gamma_1'(h(t)) \cdot h'(t) \rangle \, dt \, .$$

$h(b_2) = b_1$

$$= \int_{a_2}^{b_2} \langle g(\gamma_1(h(t))), \gamma_1'(h(t)) \rangle \, \underline{h'(t) \, dt} \overset{*}{=} \int \langle g(\gamma_1(s)), \gamma_1'(s) \rangle \, ds = \int_{\gamma_1} \langle g, \tau \rangle \, ds$$

$s = h(t)$

$h(a_2) = a_1 \quad$ se $h'(t) > 0$

se $h'(t) < 0$

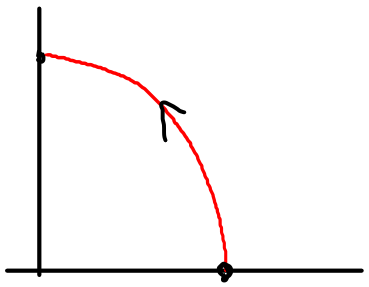
$$* = - \int_{\gamma_1} \langle g, \tau \rangle \, ds$$

Se γ_1 e γ_2 sono equiverse si ha $\int_{\gamma_1} \langle g, T\rangle \, ds = \int_{\gamma_2} \langle g, T\rangle \, ds$

" hanno verso opposto $\int_{\gamma_1} \langle g, T\rangle \, ds = -\int_{\gamma_2} \langle g, T\rangle \, ds$

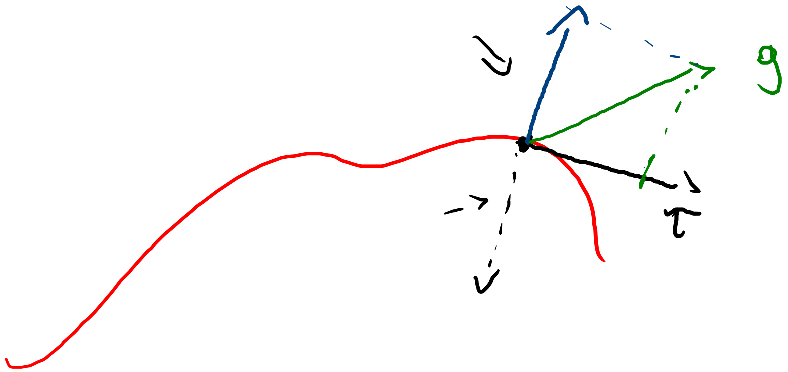
Es: Si calcoli il lavoro compiuto dal campo $g(x,y) = (x^2, -xy)^T$ su una particella che si muove lungo l'arco di cerchio $\gamma(t) = (\cos t, \operatorname{sen} t)^T$ per $t \in [0, \frac{\pi}{2}]$.

$$\gamma'(t) = (-\operatorname{sen} t, \cos t)^T$$



$$L = \int_\gamma \langle g, T\rangle \, ds = \int_0^{\pi/2} \langle (\cos^2 t, -\cos t \operatorname{sen} t)^T, (-\operatorname{sen} t, \cos t)^T \rangle \, dt =$$

$$= \int_0^{\pi/2} -2 \operatorname{sen} t \cos^2 t \, dt = \frac{2}{3} \left[\cos^3 t \right]_0^{\pi/2} = -\frac{2}{3}$$



Se $\gamma(t)$ in $\mathbb{R}^2$ $\quad \gamma(t) = (x(t), y(t))^T \qquad \gamma'(t) = (x'(t), y'(t))^T$

il vettore normale è $(y'(t), -x'(t))^T$

$(-y'(t), x'(t))^T$

definiamo vettore normale della curva piana nel punto il vettore $(y'(t), -x'(t))^T$

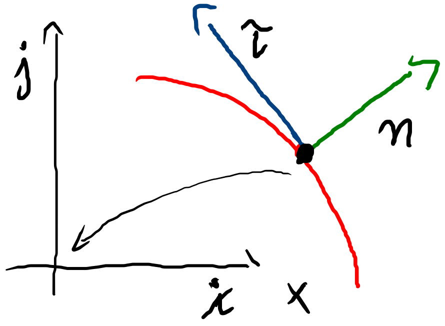
$w(t)$

vettore tangente $\quad \gamma'(t) = (x'(t), y'(t))^T$

versore tangente $\quad \tau(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$

vettore normale $\quad w(t) = (y'(t), -x'(t))^T$

versore normale $\quad \dfrac{w(t)}{\|w(t)\|} = n(t)$



Il verso "standard" del vettore normale è tale da rendere congruente il sistema di assi definito da $n(t)$ e $\tau(t)$ con gli assi x e y (nell'ordine).

Integrale di linea della componente normale del campo g: in $\mathbb{R}^2$

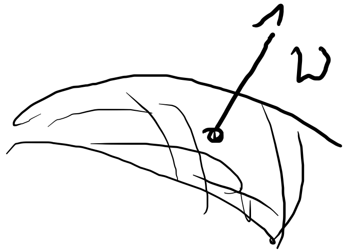
$$\int_\gamma \langle g, \nu \rangle \, ds = \int_a^b \langle g(x(t), y(t)), (y'(t), -x'(t))^T \rangle \, dt = \int_\gamma X \, dy - Y \, dx$$

$$\gamma(t) = (x(t), y(t))^T \qquad \gamma : [a,b] \to \mathbb{R}^2$$

$$g(x,y) = (X(x,y), Y(x,y))^T$$

Significato fisico? È il flusso del campo che attraversa la curva

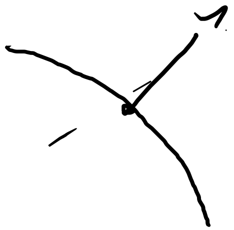
In $\mathbb{R}^3$ si studierà il flusso attraverso una superficie



$$\iint_\sigma \langle g, \nu \rangle \, d\sigma = \iint_K \langle g(\sigma(s,t)), w(s,t) \rangle \, ds\, dt$$

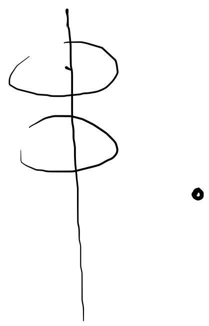
$$\frac{\partial \sigma}{\partial s} \wedge \frac{\partial \sigma}{\partial t}$$

$\sigma : K \subseteq \mathbb{R}^2 \to \mathbb{R}^3$



Biot-Savard

$$B(r) = \int_\gamma \frac{\mu_0}{4\pi} I \frac{dl \times (r - r')}{|r-r'|^3}$$



$\mathbb{R}^3$ $\quad (x,y,z)^T \quad (x_0,y_0,z_0)^T$ punto fissato

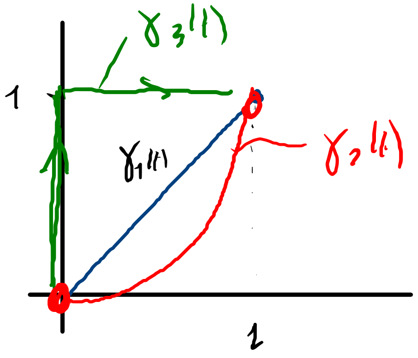
$$g(x,y,z) = \frac{(x-x_0,\, y-y_0,\, z-z_0)^T}{\|(x-x_0,\, y-y_0,\, z-z_0)^T\|^3}$$

$\langle g, a\rangle \quad \langle g, b\rangle$

$$B(x_0,y_0,z_0) = \int_\gamma \frac{\mu_0}{4\pi} I \, \underbrace{g \wedge \tau}_{g \times \tau} \, ds$$

$$\int_a^b \dots \left(g(\gamma(t)) \wedge \gamma'(t)\right) dt$$

Es. $g(x,y) = (y^2, 2xy)^T$



$\gamma_1(t) = (t,t)^T \quad t \in [0,1] \qquad \gamma_1'(t) = (1,1)^T$

$$L_1 = \int_{\gamma_1} \langle g, \tau \rangle \, ds = \int_0^1 (t^2 \cdot 1 + 2t^2 \cdot 1)\, dt = \int_0^1 3t^2\, dt = 1$$

$\gamma_2(t) = (t, t^2)^T \qquad \gamma_2'(t) = (1, 2t)^T$

$$L_2 = \int_{\gamma_2} \langle g, \tau \rangle \, ds = \int_0^1 (t^4 \cdot 1 + 2t^3 \cdot 2t)\, dt = \int_0^1 5t^4\, dt = 1$$

$\gamma_3(t) = \gamma_{3,a} + \gamma_{3,b} \qquad \gamma_{3,a}(t) = (0,t)^T \quad t \in [0,1] \qquad \gamma_{3,a}'(t) = (0,1)^T$

$\gamma_{3,b}(t) = (t,1)^T \quad t \in [0,1] \qquad \gamma_{3,b}'(t) = (1,0)^T$

$$L_3 = \int_{\gamma_3} \langle g, \tau \rangle \, ds = \int_{\gamma_{3,a}} \langle g, \tau \rangle \, ds + \int_{\gamma_{3,b}} \langle g, \tau \rangle \, ds = \int_0^1 (t^2 \cdot 0 + \underbrace{2 \cdot 0 \cdot t \cdot 1}_{=0})\, dt + \int_0^1 (1 \cdot 1 + 2t \cdot 0)\, dt = 1$$

Campi conservativi

$g: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ A aperto;

se esiste un campo scalare $U: A \to \mathbb{R}$ tale che $\nabla U = g$, U si dice un potenziale di g.

Un campo g si dice conservativo se esiste un potenziale di g.

<u>OSS</u>: Se U è un potenziale di g, allora per ogni costante c <u>$U+c$</u> è un potenziale di g. Se U_1 e U_2 sono potenziali di g, allora $U_1 - U_2$ è costante.

Sia A connesso

$\left[\nabla(U+c) = \nabla(U) \right. \qquad \nabla U_1 = \nabla U_2 \Rightarrow \nabla(U_1 - U_2) = 0 \Rightarrow U_1 - U_2$ è costante.

OSS Sia g un campo conservativo differenziabile $g = \nabla U$ o

$$g = (g_1, g_2, \dots, g_N)^T = \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \dots, \frac{\partial U}{\partial x_N} \right)^T$$ U è 2 volte differenziabile

$$\frac{\partial^2 U}{\partial x_i \partial x_j} = \frac{\partial^2 U}{\partial x_j \partial x_i} \Rightarrow \frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j}$$

$$\frac{\partial}{\partial x_i} g_j$$

Condizione necessaria affinché un campo g differenziabile sia conservativo è che si abbia

$$\forall i, j \quad \frac{\partial}{\partial x_j} g_i = \frac{\partial}{\partial x_i} g_j$$

Def: rotore di un campo $g : A \subseteq \mathbb{R}^3 \to \mathbb{R}^3$

$$g(x,y,z) = (X, Y, Z)^T$$

$$\operatorname{rot} g : A \subseteq \mathbb{R}^3 \to \mathbb{R}^3$$

$$\operatorname{rot} g(x,y,z) = \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right)^T$$

$$\frac{Z}{y} - \frac{Y}{} \quad \frac{X}{\partial z} \quad \frac{Z}{\partial x} \quad Y \quad X$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix}$$

se $N = 2$ $\quad g(x,y) = (X, Y)^T$

$$\tilde{g}(x,y,z) = \left(X(x,y), Y(x,y), 0 \right)^T$$

$$\operatorname{rot} g = \operatorname{rot} \tilde{g} = \left(0, 0, \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right)^T$$

Se g è differenziabile e conservativo, allora $\text{rot}\, g = 0$.

[Un campo si dice irrotazionale se $\text{rot}\, g$]

<u>OSSERVAZIONE</u> Sia $g : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ campo conservativo.

$\gamma : [a,b] \to \mathbb{R}^n$ $\quad \Gamma \subset A$. Allora, se $g = \nabla U$, si ha
$\quad\quad \overset{\shortparallel}{\gamma([a,b])}$

$$\int_\gamma \langle g, \tau \rangle \, ds = \int_\gamma \langle \nabla U, \tau \rangle \, ds = \int_a^b \underbrace{\langle \nabla U(\gamma(t)), \gamma'(t) \rangle}_{\overset{\shortparallel}{\frac{d}{dt}(U \circ \gamma)(t)}} \, dt = \int_a^b \frac{d}{dt}(U \circ \gamma)(t) \, dt$$

$$= U(\gamma(b)) - U(\gamma(a))$$

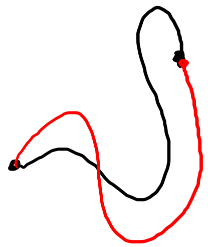
In particolare il lavoro non dipende dalla curva γ ma soltanto dal valore agli estremi.

Teorema di caratterizzazione dei campi conservativi

Sia $A \subseteq \mathbb{R}^n$ aperto connesso, $g: A \to \mathbb{R}^n$ continua. Sono equivalenti:

1) g è conservativo (cioè esiste un potenziale U di g).

2) Siano γ_1, γ_2 curve regolari a tratti t.c. che $\gamma_1(a_1) = \gamma_2(a_2)$ $\gamma_1(b_1) = \gamma_2(b_2)$



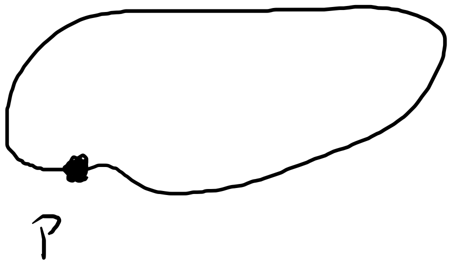
$\gamma_i : [a_i, b_i] \to A$

Allora $\int_{\gamma_1} \langle g, \tau \rangle \, ds = \int_{\gamma_2} \langle g, \tau \rangle \, ds$.

3) Per ogni γ chiusa regolare a tratti si ha $\int_\gamma \langle g, \tau \rangle \, ds = 0$

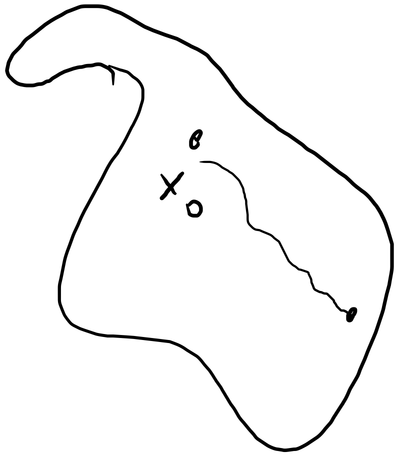
[Notazione: se γ è una curva chiusa l'integrale $\int_\gamma \langle g, \tau \rangle \, ds$ si indica con $\oint_\gamma \langle g, \tau \rangle \, ds$ e si dice circuitazione]

Dim 1) $\Rightarrow$ 2) fatto 2) $\Rightarrow$ 3) considero $\gamma_0(t) = c$



$\gamma : [a,b] \to \mathbb{R}^N$

$\gamma(a) = \gamma(b)$



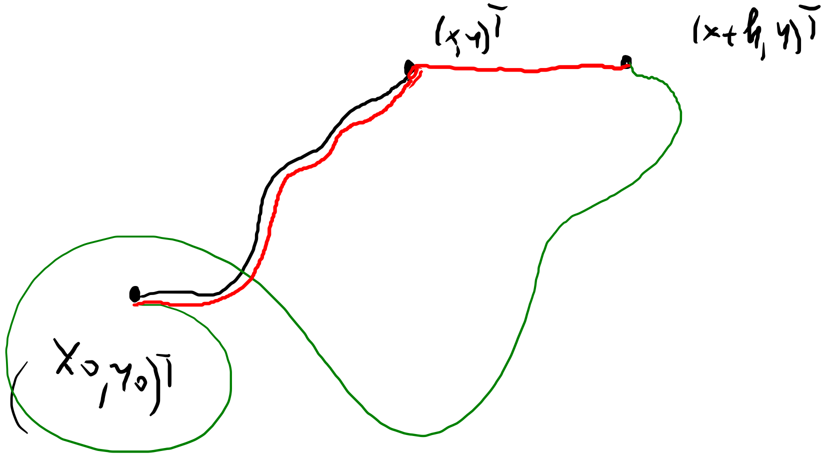
$\gamma_0(t) = \gamma(a) = \gamma(b)$ $\int_\gamma \langle g, \tau \rangle \, ds = \int_{\gamma_0} \langle g, \tau \rangle \, ds = 0$

Dimostriamo che 2) $\Rightarrow$ 1) $(N = 2)$

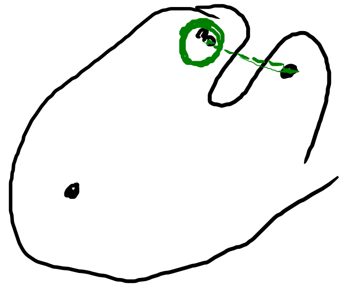
Dobbiamo definire U: fissiamo $x^\circ \in A$; per ogni $x \in A$ sia γ una curva congiungente x_0 a x; poniamo $U(x) = \int_\gamma \langle g, \tau \rangle \, ds$

U è ben definita per ipotesi 2); dimostriamo che $\nabla U = g$.

$$\frac{\partial U}{\partial x} = \lim_{h \to 0} \frac{1}{h} \left[U(x+h, y) - U(x,y) \right]$$



$$U(x,y) = \int_{\gamma_{(x_0,y_0)^T \to (x,y)^T}} \langle g, \tau \rangle \, ds$$



$$U(x+h,y) = \int_{\gamma_{(x_0,y_0)^T \to (x+h,y)^T}}$$

Come curva congiungente $(x_0,y_0)^T$ e $(x+h,y)^T$ scegliamo la curva $\underline{\underline{\gamma}} = \gamma_1 + \gamma_2$ dove γ_1 è la curva congiungente $(x_0,y_0)^T$ e $(x,y)^T$

e $\gamma_2 : [0,1] \to \mathbb{R}^2 \quad \gamma_2(t) = (x+th, y)^T$

$$\frac{\partial U}{\partial x} = \lim_{h \to 0} \frac{1}{h} \left[\int_{\gamma} \langle g, \tau \rangle \, ds - \cancel{\int_{\gamma_1} \langle g, \tau \rangle \, ds} \right] = \lim_{h \to 0} \frac{1}{h} \int_{\gamma_2} \langle \overset{(X,\,Y)}{g}, \tau \rangle \, ds =$$

$$\gamma = \gamma_1 + \gamma_2$$

$$\parallel$$

$$\cancel{\int_{\gamma_1} \langle g, \tau \rangle \, ds} + \int_{\gamma_2} \langle g, \tau \rangle \, ds$$

$$\gamma_2(t) = (x + th, y)^T$$

$$\gamma_2'(t) = (h, 0)^T$$

$$= \lim_{h \to 0} \cancel{\frac{1}{h}} \int_0^1 \left(X(x + th, y) \cdot \cancel{h} + Y(x + th, y) \cdot 0 \right) dt = \lim_{h \to 0} \underbrace{\int_0^1 X(x + th, y) \, dt}_{X(x + \overset{\to 0}{\varepsilon}, y) \cdot 1}$$

$$= X(x, y)$$

Si verifica nello stesso modo che $\frac{\partial U}{\partial y} = Y(x, y)$ quindi $\nabla U = g$.

Esempi di calcolo del potenziale

$g(x,y) = (y^2, \boxed{2xy + y^2 + 1})^T$ $\quad U(x,y) = ?$ $\quad \int_\gamma \langle g, \tau \rangle \, ds$

$g(x,y) = (X(x,y), Y(x,y))^T$ $\quad \dfrac{\partial U}{\partial x} = X$ $\quad \dfrac{\partial U}{\partial y} = Y$

$\dfrac{\partial U}{\partial x} = y^2$ $\quad U(x,y) = \int X(x,y)\,dx + h(y)$

$\int \dfrac{\partial U}{\partial x}\,dx = \int y^2\,dx = y^2 x + h(y)$ $\quad \boxed{U(x,y) = y^2 x + \frac{1}{3}y^3 + y}$

$\| $

$U(x,y)$

$\dfrac{\partial U}{\partial y} = 2xy + y^2 + 1 \Rightarrow \cancel{2yx} + h'(y) = \cancel{2xy} + y^2 + 1$

$\Rightarrow h'(y) = y^2 + 1 \quad \int h'(y)\,dy = \int (y^2+1)\,dy = \frac{1}{3}y^3 + y$

$g(x,y) = (y, -x)^T$ $\qquad U(x,y) = ?$

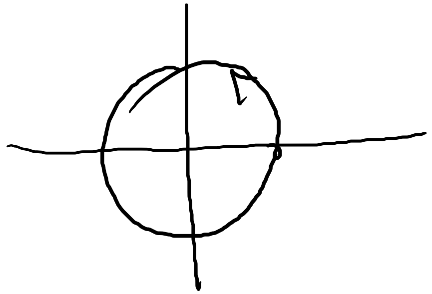
$U = \int y \, dx = xy + h(y)$

$\frac{\partial U}{\partial y} = \underbrace{x + h'(y)}_{} = \underbrace{-x}_{}$?? $\boxed{-2x = h'(y)}$

Il campo non è conservativo! $\qquad \mathrm{rot}\, g = (0, 0, -2)^T \neq (0,0,0)^T$

Es: $g(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)^T$ $g: \underbrace{\mathbb{R}^2 \setminus \{(0,0)^T\}}_{A}$

"X" "Y"



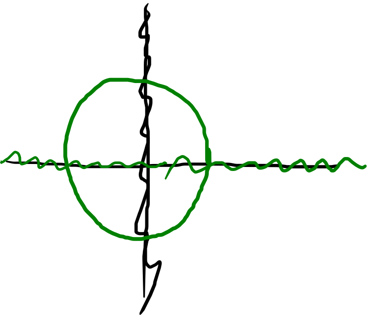
$U(x,y) = \operatorname{arctg}\left(\frac{y}{x}\right)$ — $U: \mathbb{R}^2 \setminus \{x=0\}$

$$\frac{\partial U}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{x^2}{x^2+y^2} \cdot \frac{y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial U}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$ OK

$\gamma(t) = (\cos t, \operatorname{sen} t)^T$ $\gamma'(t) = (-\operatorname{sen} t, \cos t)^T$

$$\oint_\gamma \langle g, \tau \rangle \, ds = \int_0^{2\pi} [-\operatorname{sen} t \cdot (-\operatorname{sen} t)] + \cos t \cdot \cos t) \, dt = \int_0^{2\pi} 1 \, dt = 2\pi \neq 0!$$



$U(x,y) = -\operatorname{arctg}\left(\frac{x}{y}\right)$ $\frac{\partial U}{\partial x} = X$ $\frac{\partial U}{\partial y} = Y$

NO TUTORATO

OGGI