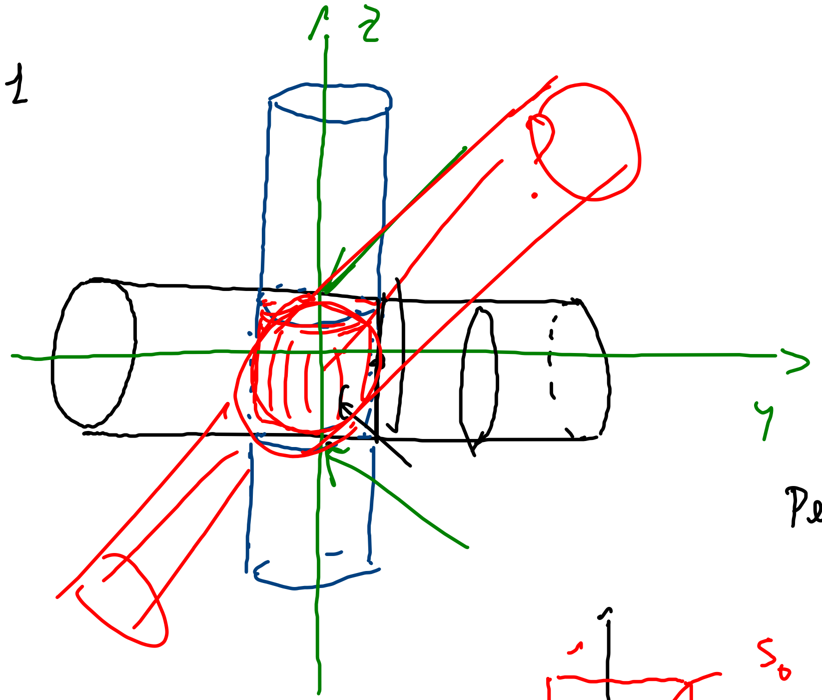
$R = 1$



$$\boxed{\text{Es. } 19}$$

$$\boxed{\begin{aligned} x^2+y^2 &\le 1 \\ x^2+z^2 &\le 1 \end{aligned}}$$

~~$S_y = \{(x,z)^T : (x,y,z)^T \in S\}$~~

Per sezioni rispetto all'asse x

$$\int_{m\,-1}^{M\,1} \left(\iint_{S_x} dy\,dz \right) dx$$

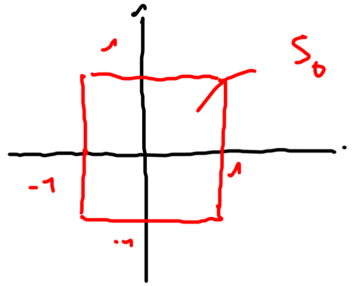
Numero!

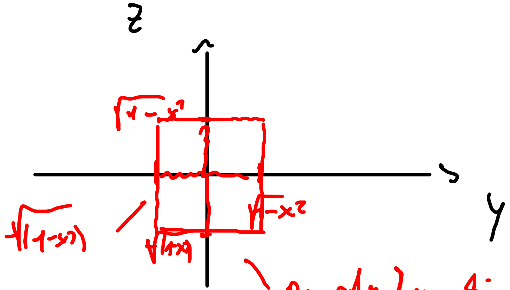
Fisso x $\quad S_x = \{(y,z)^T \in \mathbb{R}^2 : \; y^2 \le \underbrace{1-x^2}, \; z^2 \le \underbrace{1-x^2}\}$

Nel piano yz $\quad x=0$

$y^2 \le 1$

$z^2 \le 1$





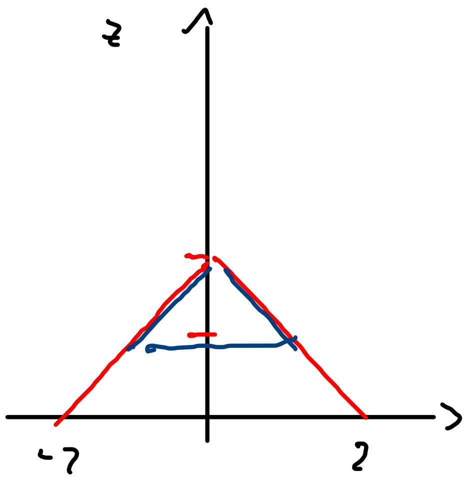
quadrato di lato $2\sqrt{1-x^2}$

Area $S_x = 4(1-x^2)$

$$V = \int_{-1}^{1} 4(1-x^2)\,dx = \; = 4\left[x-\frac{1}{3}x^3\right]_{-1}^{1} = 4\left(1-\frac{1}{3}+1-\frac{1}{3}\right) = 8\left(1-\frac{1}{3}\right) = \frac{16}{3}$$

$z \in [1,2]$ $\quad z \leq 2 - \sqrt{x^2+y^2}$

$y=0$



$z = 2 - |x|$

$\rho = 2 - z$

i) $\delta = z^2$

proporzionale a $d^2\left((x,y,z)^T, (x,y,0)^T\right)$

$(x-x)^2 + (y-y)^2 + (z-0)^2 = z^2$

ii) $\delta = \dfrac{1}{z^2}$

$\sqrt{x^2+y^2} \leq 2-z$

$191 \cdot 3 = 573$ $\qquad 280 \cdot 2 = 560$

$63 \cdot 5 = 315$ $\quad 315 - 124 = 191$

Per sezioni

$$\int_1^2 \left(\iint_{S_z} z^2 \, dx\,dy \right) dz = \int_1^2 z^2 \pi \left(4 - 4z + z^2\right) dz = \pi \left(\frac{1}{5} z^5 - z^4 + \frac{4}{3} z^3 \right)_1^2$$

$z^4 - 4z^3 + 4z^2$

$S_z = \{ (x,y)^T \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leq 2 - z \}$ $\qquad$ Area $S_z = \pi (2-z)^2$

$$\frac{1}{5} 2^5 - 2^4 + \frac{4}{3} 2^3 - \frac{1}{5} + \frac{1}{4} - \frac{4}{3}$$

$$\frac{31}{5} - 16 + \frac{32}{3} + \frac{1}{4} \cancel{\frac{1}{5}} - \frac{4}{3} = \frac{31}{5} - \frac{63}{4} + \frac{28}{3}, \quad \frac{124 - 315}{20} + \frac{28}{3} \qquad -\frac{191}{20} + \frac{28}{3} = \frac{-573 + 560}{}$$

2

1

1 2

$$\int_1^2 \pi (2-z)^2 \cancel{z^2}\,\frac{1}{z^2}\, dz$$

$$(4-4z+z^2)\,\cancel{z^2}\,\frac{1}{z^2} = \cancel{4z^2-4z^3+z^4}$$

$$\cancel{\left[\frac{4}{3}z^3 - z^4 + \frac{1}{5}z^5\right]_1^2}$$

$$\frac{4}{3}2^3 - 2^4 + \frac{1}{5}2^5 - \frac{4}{3} + 1 - \frac{1}{5} =$$

$$\frac{4}{3}(2^3-1) + (1-2^4) + \frac{1}{5}(2^5-1)$$

$$\frac{28}{3} \quad -15 \quad + \frac{1}{5}31$$

$$\pi\,\frac{8}{15}$$

$$\frac{140+93}{15}$$

$$\frac{233}{15} - \frac{225}{15} = \frac{8}{15}$$

$$\pi\int_1^2 \left(\frac{4}{z^2} - \frac{4}{z} + 1\right)$$

$$\left[-\frac{1}{3}\frac{4}{z^3} - 4\log z + z\right]_1^2$$

15 ·
15 =
75
15
225

140 +
93 =
133

Integrali generalizzati

A_n misurabile

$f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ $(A_n)_n$ successione invadente adatta a f se $A_n \subseteq A_{n+1} \subseteq E$

per ogni misurabile $M \subseteq E$ $\lim_{n \to +\infty} m(M - A_n) = 0$

$\exists \int_{A_n} f \, dm$

Idea $\int_E f \, dm = \lim_{n \to +\infty} \int_{A_n} f \, dm$ NON SI PUÒ

l'integrale può dipendere dalla successione scelta.

Teorema

Sia $E \subseteq \mathbb{R}^N$ $f: E \to \mathbb{R}$ $f(x) \geq 0$ $\forall x \in E$. Siano $(A_n)_n$ $(B_n)_n$ successioni invadenti di E adatte ad f.

[si osservi che essendo $f \geq 0$ e $\overset{B_n}{A_n} \subseteq \overset{B_{n+1}}{A_{n+1}}$ si ha che la successione $\left(\int_{\underset{B_n}{A_n}} f\,dm \right)_n$ è crescente; quindi esiste sempre $\lim_{n \to +\infty}$]

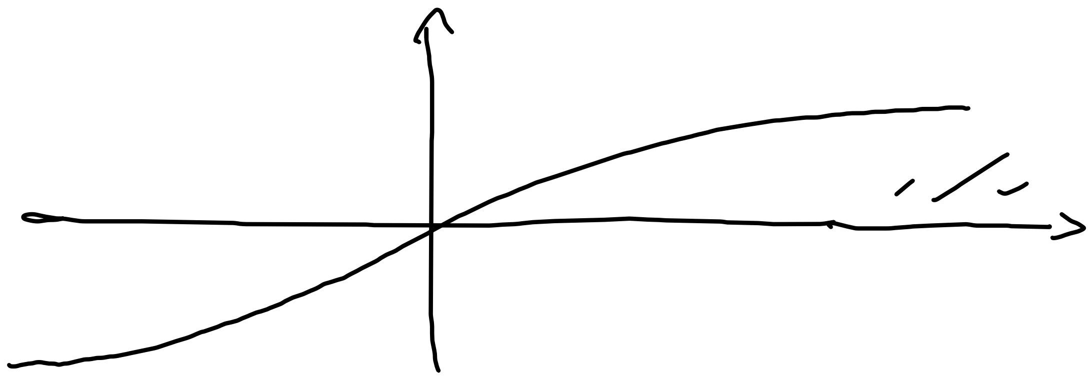
Allora $$\lim_{n \to +\infty} \int_{A_n} f\,dm = \lim_{n \to +\infty} \int_{B_n} f\,dm$$

Definizione Se $f: E \subseteq \mathbb{R}^N \to \mathbb{R}$ $f(x) \geq 0$ $\forall x \in E$. Diremo che f è integrabile in senso generalizzato su E se, presa comunque una successione invadente adatta a f il limite $\lim_{n \to +\infty} \int_{A_n} f\,dm$ esiste finito. Questo limite si indica $\int_E f\,dm$.

[OSS: se $f(x) \leq 0$ si considera $-f(x)$ e si definisce $\int_E f\,dm := -\int_E -f\,dm$]

$E = E_1 \cup E_2$ $\quad m(E_1 \cap E_2) = \emptyset$ $\quad \int_{E_1 \cup E_2} f\,dm = \int_{E_1} f\,dm + \int_{E_2} f\,dm$

Es $\quad f(x) = \operatorname{arctg} x$



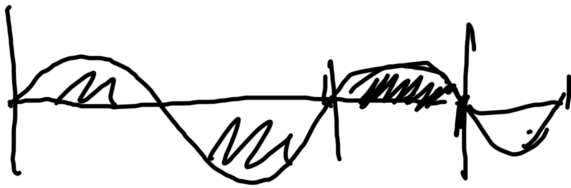
$$\int_{\mathbb{R}} \operatorname{arctg} x \, dx = \int_{]-\infty,0]} \operatorname{arctg} x \, dx + \int_{]0,+\infty[} \operatorname{arctg} x \, dx$$ non ha significato

non esistono!

$$\int_0^{+\infty} \operatorname{sen} x \, dx$$

$E = [0, +\infty[$

$$A_n = [0, 2n\pi] \qquad B_n = [0, (2n+1)\pi]$$

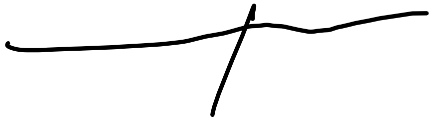


$$\int_{A_n} \operatorname{sen} x \, dx = 0 \quad \forall n \qquad \int_{B_n} \operatorname{sen} x \, dx = \int_0^{\pi} \operatorname{sen} x \, dx = 2$$

$$\lim_{n \to +\infty} \int_{A_n} \operatorname{sen} x \, dx = 0 \qquad \lim_{n \to +\infty} \int_{B_n} \operatorname{sen} x \, dx = 2$$

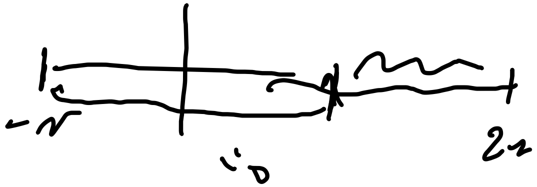
$E = \mathbb{R}$

$$f(x) = \frac{2x}{1+x^2}$$



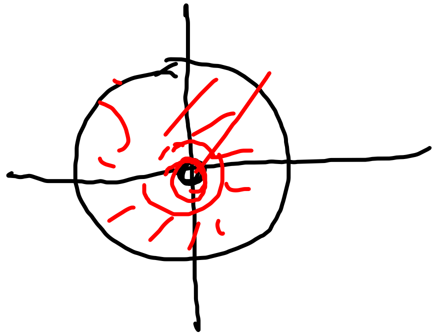
$\log(1+x^2)$

$$A_n = [-n, n] \qquad B_n = [-n, 2n]$$



$$\int_{A_n} f(x)\,dx = 0 \qquad \int_{B_n} f(x)\,dx = \log 4$$

Es: $$\iint_{\underbrace{B((0,0)^T,1)\setminus\{(0,0)^T\}}_{=E}} \log\left(\frac{1}{x^2+y^2}\right) dx\,dy$$



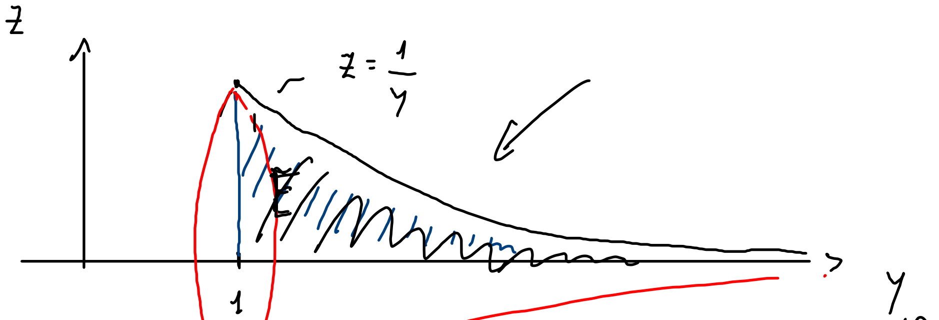
$A_n := B(0,1)\setminus B(0,\frac{1}{n})$ $\qquad A_n \subset A_{n+1} \qquad m(E\setminus A_n) \to 0$

$$\iint_{A_n} \log\left(\frac{1}{x^2+y^2}\right) dx\,dy = \int_0^{2\pi}\left(\int_{1/n}^{1} -\log(\rho^2)\cdot\rho\, d\rho\right) d\vartheta = -2\pi\int_{1/n}^{1} 2\log(\rho)\,\rho\, d\rho =$$

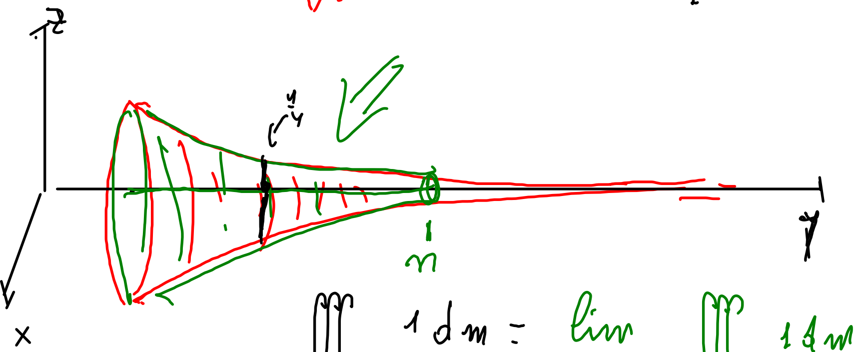
$$= -4\pi\left(\left[\frac{1}{2}\rho^2\cdot\log\rho\right]_{1/n}^{1} - \underbrace{\int \frac{1}{2}\rho^{\cancel{2}}\frac{1}{\cancel{\rho}}\, d\rho}_{-\frac{1}{2}\cdot\frac{1}{2}\rho^2}\right) = -4\pi\left(\underbrace{-\frac{1}{2}\frac{1}{n^2}\log\left(\frac{1}{n}\right)}_{\to 0} - \frac{1}{4} + \underbrace{\frac{1}{4}\frac{1}{n^2}}_{\to 0}\right)$$

$$\lim_{n\to+\infty}\iint_{A_n} f(x,y)\,dx\,dy = \pi$$

Es: $f(x) = 1$ $\int_E dm$ si parla di misura generalizzata (area / volume)



L'area generalizzata di E è $\iint_E 1\,dx\,dy = \int_1^{+\infty} \frac{1}{y}\,dy = +\infty$



Volume del solido ottenuto ruotando E attorno di asse y

$A_n = \{(x,y,z)^T \in S : y \in [1,n]\}$

$$\iiint_S 1\,dm = \lim_{n\to+\infty} \iiint_{A_n} 1\,dm$$

$$\iiint_{A_n} dm = \int_1^n \left(\iint_{S_y} 1\,dx\,dz \right) dy = \int_1^n \pi \frac{1}{y^2}\,dy = \left[\pi\left(-\frac{1}{y}\right)\right]_1^n = \pi\left(1 - \frac{1}{n}\right)$$

per sezioni rispetto asse y $y \in [1,n]$ $S_y = \{(x,z)^T \in \mathbb{R}^2 : x^2 + z^2 \le \frac{1}{y^2}\}$

Area di $S_y = \pi \frac{1}{y^2}$

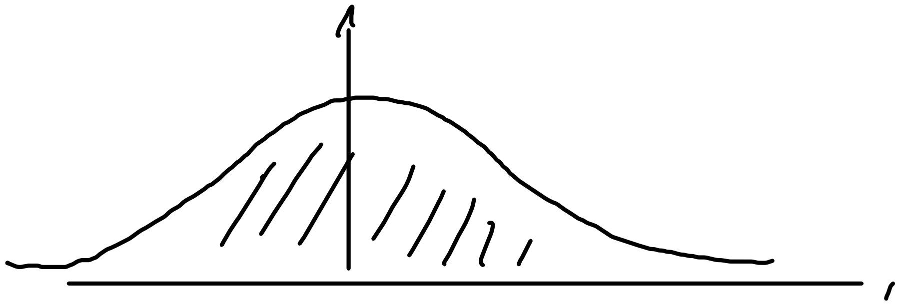
$$\lim_{n\to+\infty} \iiint_{A_n} 1\,dm = \pi$$

Es. importante: l'integrale di Gauss

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx$$

infinitesimo

di ordine superiore a $\pm\infty$. quant'è?



$u = -\rho^2$

$du = -2\rho\, d\rho$

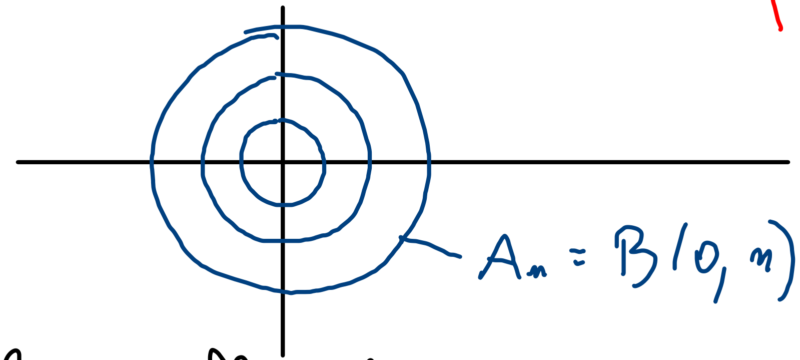
Calcoleremo $$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$$

$$\iint_{A_n} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\left(\int_0^{n} e^{-\rho^2}\cdot\rho\,d\rho\right)d\vartheta =$$

in polari $x = \rho\cos\vartheta$ $\quad \vartheta\in[0,2\pi]$

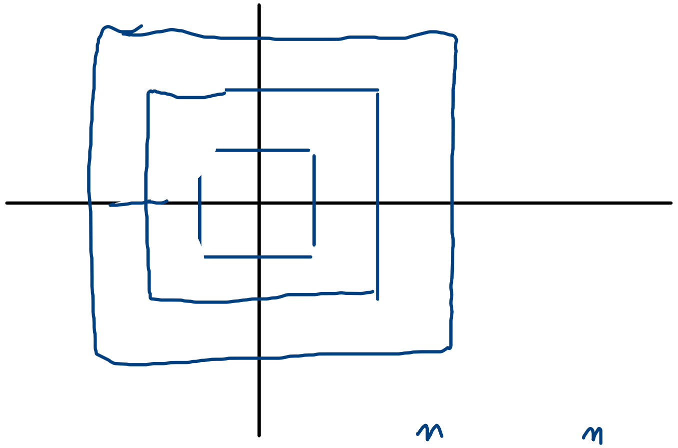
$y = \rho\sin\vartheta$ $\quad \rho\in[0,n]$

$$= \int_0^{2\pi}\left[-\frac{1}{2}e^{-\rho^2}\right]_0^{n} d\vartheta = \cancel{2}\pi\,\frac{1}{\cancel{2}}\left(1-e^{-n^2}\right)$$



$A_n = B(0,n)$

$$\lim_{n\to+\infty}\iint_{A_n} f(x,y)\,dx\,dy = \pi$$



$$B_n = [-n, n] \times [-n, n]$$

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \underbrace{\lim_{n \to +\infty} \iint_{B_n} e^{-x^2-y^2}\,dx\,dy}$$

$$\|$$

$$\pi$$

$$\iint_{B_n} e^{-x^2-y^2}\,dx\,dy = \int_{-n}^{n} \left(\int_{-n}^{n} e^{-x^2} \cdot e^{-y^2}\,dy \right) dx = \int_{-n}^{n} e^{-x^2} \underbrace{\left(\int_{-n}^{n} e^{-y^2}\,dy \right)} dx =$$

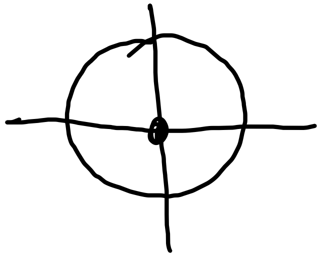
$$= \int_{-n}^{n} e^{-y^2}\,dy \cdot \int_{-n}^{n} e^{-x^2}\,dx = \left[\int_{-n}^{n} e^{-x^2}\,dx \right]^2$$

$$\pi = \lim_{n \to +\infty} \left[\int_{-n}^{n} e^{-x^2}\,dx \right]^2 = \left[\int_{-\infty}^{+\infty} e^{-x^2}\,dx \right]^2$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Esempi importanti

Studiamo $\dfrac{1}{\|x\|^\alpha}$



$$B_n = B(0,1) - B\left(0,\tfrac{1}{n}\right)$$

$n = 2$

$$\iint_{B(\underset{(0,0)^T}{0},1) \setminus \{(0,0)^T\}} \frac{1}{(\sqrt{x^2+y^2})^\alpha}\, dx\,dy = \lim_{n\to+\infty} \iint_{B_n} (x^2+y^2)^{-\alpha/2}\, dx\,dy =$$

$$= \lim_{n\to+\infty} \int_0^{2\pi} \left(\int_{1/n}^{1} \underbrace{(\rho^2)^{-\alpha/2}\cdot \rho}_{\rho^{1-\alpha}}\, d\rho \right) d\vartheta = \lim_{n\to+\infty} 2\pi \left[\frac{1}{2-\alpha}\, \rho^{2-\alpha} \right]_{1/n}^{1} =$$

$\alpha \neq 2$

$$= \frac{2\pi}{2-\alpha} \cdot \left(1 - \lim_{n\to+\infty} \frac{1}{n^{2-\alpha}} \right) = \begin{cases} \dfrac{2\pi}{2-\alpha} & \text{se } \alpha < 2 \\ +\infty & \text{se } \alpha > 2 \end{cases}$$

Teorema Criterio dell'ordine di infinito per l'integrale generalizzato

$f : B(0,R) \setminus \{0\} \subseteq \mathbb{R}^N \to \mathbb{R}$ $\quad f(x) \geq 0 \quad \forall x \; ; \quad \lim_{x \to 0} f(x) = +\infty$

$\text{Ord.}\, f \leq \alpha \quad \alpha \in \mathbb{R} \quad \alpha < N.$

f integrabile su ogni corona circolare $B(0,R) \setminus B(0,\varepsilon)$ $\quad \varepsilon > 0.$

Allora f è integrabile in senso generalizzato su $B(0,R)$.

Dim per confronto con $\frac{1}{\|x\|^\alpha}$

$N = 3 \quad \iiint_B \frac{1}{\|x\|^\alpha}\, dm = \quad \left(\sqrt{x^2+y^2+z^2}\right)^\alpha$

$x = \rho \operatorname{sen}\varphi \cos\vartheta \quad y = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \quad z = \rho\cos\varphi$

$$\int_{\frac{1}{n}}^{1} \left(\int_0^{\pi} \left(\int_0^{2\pi} \frac{1}{\rho^\alpha} \cdot \rho^2 \operatorname{sen}\varphi \, d\vartheta \right) d\varphi \right) d\rho =$$

$\rho^{2-\alpha}$

$\int_0^{\pi} \operatorname{sen}\varphi \, d\varphi = 2$

$$= 2\pi \cdot 2 \cdot \left[\frac{1}{3-\alpha} \rho^{3-\alpha} \right]_{\frac{1}{n}}^{1}$$

se $\alpha < 3$ finito

$\alpha > 3$ $+\infty$