- Curve rettificabili

Due osservazioni sugli integrali vettoriali

Sia $f: E \subseteq \mathbb{R}^N \to \mathbb{R}^M$ $\quad f = (f_1, f_2, \dots, f_M)^T$

$$\int_E f\, dx = \left(\int_E f_1\, dx, \dots, \int_E f_M\, dx \right)^T$$

In particolare se $\gamma: I \to \mathbb{R}^N$ è una curva $\quad \gamma(t) = (x(t), y(t), z(t))^T$

$$\int_I \gamma(t)\, dt = \left(\int_I x(t)\, dt, \int_I y(t)\, dt, \int_I z(t)\, dt \right)^T$$

OSS. 1: Formula di Torricelli-Barrow $\quad t_1, t_2 \quad \gamma \in C^1$

$$\gamma(t_2) - \gamma(t_1) = \int_{t_1}^{t_2} \gamma'(\xi)\, d\xi \qquad (x(t_2) - x(t_1), y(t_2) - y(t_1))^T = \left(\int_{t_1}^{t_2} x'(\xi)\, d\xi, \int_{t_1}^{t_2} y'(\xi)\, d\xi \right)^T$$

$\gamma = (x, y)^T \qquad N = 2$

Oss. 2: Disuguaglianza sulla norma dell'integrale

Sia $\gamma : I \to \mathbb{R}^n$, allora $\left\| \int_I \gamma \, dt \right\| \leq \int_I \|\gamma\| \, dt$

Dim: Fissiamo un vettore $v \in \mathbb{R}^n$

$$\left\langle \int_I \gamma(t)\, dt, v \right\rangle = \int_I \langle \gamma(t), v \rangle \, dt \overset{*}{=} \qquad \leq \|\gamma(t)\| \cdot \|v\|$$

$n = 2 \quad \gamma(t) = (x(t), y(t))^T \quad v = (v_1, v_2)^T \quad \left\langle \int_I \gamma(t)\, dt, v \right\rangle = \int_I x(t)\, dt \cdot v_1 + \int_I y(t)\, dt \cdot v_2 =$

$= \int_I \left(x(t) \cdot v_1 + y(t) \cdot v_2 \right) dt = \int_I \langle \gamma(t), v \rangle \, dt$

$$* \leq \int_I \|\gamma(t)\| \cdot \underset{\text{costante}}{\|v\|} \, dt = \|v\| \int_I \|\gamma(t)\| \, dt$$

Prendiamo $v = \int_I \gamma(t)\, dt \Rightarrow \left\| \int_I \gamma(t)\, dt \right\|^{\cancel{2}} \leq \cancel{\left\| \int_I \gamma(t)\, dt \right\|} \cdot \int_I \|\gamma(t)\| \, dt$

$$\Rightarrow \left\| \int_I \gamma(t)\, dt \right\| \leq \int_I \|\gamma(t)\| \, dt$$

Dim. del Teorema di rettificabilità $\left[\gamma \in C^1(I, \mathbb{R}^n) \Rightarrow \boxed{\ell(\gamma)} \leq \int_I \|\gamma'(t)\| \, dt \right]$

Sia $\delta \in \Delta(I)$ $\quad \ell(\pi(\delta)) = \sum_{k=1}^{n} \|\gamma(t_k) - \gamma(t_{k-1})\| =$ (Torricelli-Barrow)

$$= \sum_{k=1}^{n} \left\| \int_{t_{k-1}}^{t_k} \gamma'(\xi) \, d\xi \right\| \leq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\gamma'(\xi)\| \, d\xi = \underbrace{\int_a^b \|\gamma'(\xi)\| \, d\xi}_{} \;\; K$$

$\uparrow$ disug. sulla norma

$$\int_{t_0}^{t_1} + \int_{t_1}^{t_2} + \cdots + \int_{t_{n-1}}^{t_n} = \int_{t_0}^{t_n}$$

$I = [a, b]$

$\forall \delta \quad \ell(\pi(\delta)) \leq K$

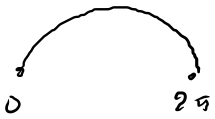
$$\boxed{\ell(\gamma) = \sup_{\delta \in \Delta(I)} \ell(\pi(\delta)) \leq K}$$

Si dimostra che $\ell(\gamma) = \int_I \|\gamma'(t)\|\,dt$

Es: lunghezza della circonferenza $\gamma(t) = (R\cos t, R\,\mathrm{sen}\,t)^T$ $\quad t\in[0,2\pi]$ $\quad \gamma'(t) = (-R\,\mathrm{sen}\,t, R\cos t)^T$

$$\ell(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|\,dt = \int_0^{2\pi}\sqrt{\underbrace{(-R\,\mathrm{sen}\,t)^2+(R\cos t)^2}_{R^2}}\,dt = \int_0^{2\pi} R\,dt = 2\pi R$$

Es: la cicloide. $\gamma(t) = (R(t-\mathrm{sen}\,t), R(1-\cos t))^T$ $\quad t\in[0,2\pi]$



$$\gamma'(t) = (R(1-\cos t), R\,\mathrm{sen}\,t)^T$$

$$\ell(\gamma) = \int_0^{2\pi}\sqrt{R^2(1-\cos t)^2 + R^2\,\mathrm{sen}^2 t}\,dt = R\int_0^{2\pi}\sqrt{2-2\cos t}\,dt = \sqrt{2}R\int_0^{2\pi}\sqrt{1-\cos t}\,dt = 2R\int_0^{2\pi}|\mathrm{sen}(t/2)|\,dt =$$

$1+\cos^2 t-2\cos t+\mathrm{sen}^2 t$

$\sqrt{2\,\mathrm{sen}^2(t/2)}$

$\cos(2x) = \cos^2(x)-\mathrm{sen}^2(x) = 1-2\,\mathrm{sen}^2(x)$

$1-\cos(2x) = 2\,\mathrm{sen}^2(x)$ $\quad t=2x$

$t/2 = u$

$dt = 2\,du$

$$= 4R\int_0^{\pi}\cancel{|}\mathrm{sen}\,u\cancel{|}\,du = 8R$$

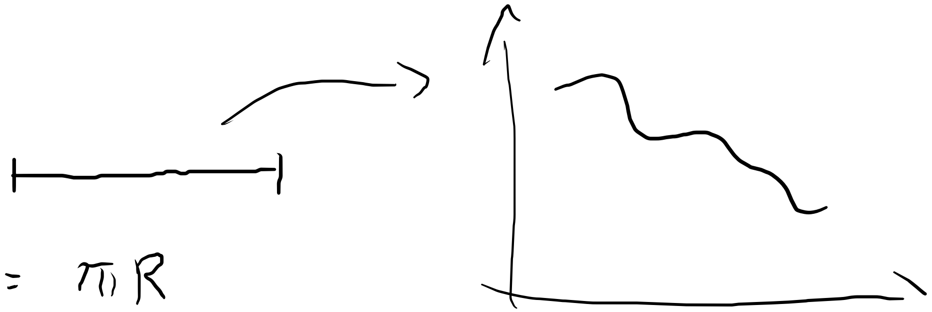
OSS Siano γ_1 e γ_2 curve C^1 equivalenti. Allora $\ell(\gamma_1) = \ell(\gamma_2)$

Infatti $\gamma_2(t) = (\gamma_1 \circ h)(t)$ $\quad h : [a_2, b_2] \to [a_1, b_1]$ $\quad C^1$ diffeom.

$$\ell(\gamma_2) = \int_{[a_2,b_2]} \|\gamma_2'(t)\|\, dt = \int_{[a_2,b_2]} \|\gamma_1'(h(t)) \cdot h'(t)\|\, dt = \int_{[a_2,b_2]} \|\gamma_1'(h(t))\| \cdot |h'(t)|\, dt =$$

$h(t) = u$

$$= \int_{[a_1,b_1]} \|\gamma_1'(u)\|\, du = \ell(\gamma_1)$$

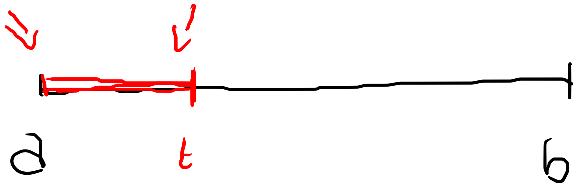


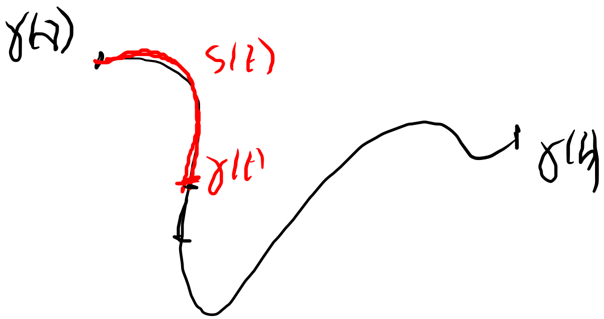
$\gamma(t) = (R\cos t, R\sin t)^T \quad t \in [0, 2\pi] \quad \ell(\gamma) = \pi R$

$\varphi(t) = (R\cos(2\pi t), R\sin(2\pi t))^T \quad t \in [0,1] \quad \ell(\varphi) = 2\pi R$

Sia $\gamma : [a,b] \to \mathbb{R}^n$ C^1 γ regolare

Sia $L = \ell(\gamma)$ poniamo $s(t)$ = lunghezza dell'arco di curva tra $\gamma(a)$ e $\gamma(t)$





$s : [a,b] \to [0,L]$

$s(t) = \displaystyle\int_a^t \|\gamma'(\tau)\|\, d\tau$ = lunghezza dell'arco di curva tra $\gamma(a)$ e $\gamma(t)$.

Si osservi che $s(t)$ è una funzione ($h(t)$) "buona" come trasformazione di coordinate;

$s'(t) = \|\gamma'(t)\|$ è continua, quindi $s \in C^1([a,b],[0,L])$

$s(t)$ è invertibile? Sì, è strettamente crescente. $s'(t) > 0$ $\forall t$ se γ è regolare

anche l'inversa è C^1.

La funzione s si dice ascisse curvilinea di γ e la curva si può parametrizzare utilizzando s; diremo allora che la curva è parametrizzata in lunghezza d'arco

$\gamma(t)$ $\quad \gamma : [a,b] \to \mathbb{R}^n$

$s : [a,b] \to [0,L]$ $\quad L = \ell(\gamma)$ $\quad$ sia $t : [0,L] \to [a,b]$ $\quad t = t(s)$ l'inversa di $s(t)$

$\varphi(s) = \gamma(t(s))$

osserviamo che
$$\varphi'(s) = \gamma'(t(s)) \cdot t'(s)$$
$$t'(s) = \frac{1}{s'(t)} \quad \leftarrow \text{Teorema della derivata della funzione inversa}$$
$$s'(t) = \|\gamma'(t)\|$$

$$\boxed{\varphi'(s) = \gamma'(t(s)) \cdot \frac{1}{\|\gamma'(t(s))\|}}$$

quindi $\boxed{\|\varphi'(s)\| = 1}$

Se la curva è parametrizzata in lunghezza d'arco $\ell(\varphi) = L$

$$\ell(\varphi) = \int_0^L \underbrace{\|\varphi'(s)\|}_{=1} ds = \int_0^L ds = L$$

Es: $\gamma(t) = (R\cos t,\ R \operatorname{sen} t)^T \quad t \in [0, 2\pi]$ $\qquad \gamma'(t) = (-R \operatorname{sen} t,\ R\cos t)^T$

parametrizziamo in lunghezza d'arco $\qquad \|\gamma'(t)\| = R$

$$s(t) = \int_0^t \|\gamma'(\xi)\|\, d\xi = Rt$$

$$t(s) = \frac{s}{R} \qquad \varphi(s) = \gamma\left(\frac{s}{R}\right) = \left(R\cos\left(\frac{s}{R}\right),\ R \operatorname{sen}\left(\frac{s}{R}\right)\right)^T$$

$$\varphi : [0, 2\pi R] \to \mathbb{R}^2$$

massa "lineare" di un filo

densità di massa $f(x,y,z)$ $\quad (x,y,z)^T = \gamma(t)$

Massa del filo? Integrale di linea

Integrali di linea di campi scalari

$\gamma : I \to \mathbb{R}^n$ curva regolare, $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ $\quad \Gamma \subseteq E$ $\quad (\Gamma = \gamma(I))$

Diremo integrale di linea di f su γ, se esiste, il numero

$$\boxed{\int_\gamma f \, ds := \int_I f(\gamma(t)) \cdot \|\gamma'(t)\| \, dt}$$

(se γ è regolare a tratti si può estendere la definizione per additività)

<u>OSS</u>: Siano γ_1 e γ_2 curve equivalenti. Allora $\int_{\gamma_1} f\,ds = \int_{\gamma_2} f\,ds$

$\gamma_2 : [a_2, b_2] \to \mathbb{R}^n$, $[a_2,b_2] = I_2$

$\gamma_1 : [a_1, b_1] \to \mathbb{R}^n$, $[a_1,b_1] = I_1$

Infatti, sia $\gamma_2(t) = (\gamma_1 \circ h)(t) \qquad h \in C^1(I_2, I_1)$

$$\int_{\gamma_2} f\,ds = \int_{a_2}^{b_2} f(\gamma_2(t)) \cdot \|\gamma_2'(t)\|\,dt =$$

$$= \int_{[a_2,b_2]} f(\gamma_1(h(t))) \cdot \|\gamma_1'(h(t))\|\,|h'(t)|\,dt = \int_{[a_1,b_1]} f(\gamma_1(u)) \cdot \|\gamma_1'(u)\|\,du$$

$u = h(t)$

$$= \int_{\gamma_1} f\,ds$$

Si osservi in particolare che, se $\varphi(s)$ è la parametrizzazione di γ in lunghezza d'arco, si ha

$$\int_{\varphi} f\,ds = \int_0^{\ell(\varphi)} f(\varphi(s)) \cdot \underbrace{\|\varphi'(s)\|}_{=1}\,ds = \int_0^{\ell(\varphi)} f(\varphi(s))\,ds$$

Se $f \equiv 1$ $\quad \int_\gamma 1\, ds = \int_\gamma ds = \ell(\gamma)$

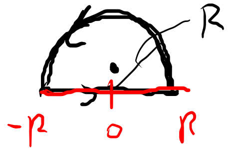
f densità

Massa lineare $\quad$ Baricentro $\quad \hat{x} = \frac{1}{M} \int_\gamma x f\, ds \quad \hat{y} = \dots \quad \hat{z} = \dots$

Momenti d'inerzia $\quad I_K = \int_\gamma d^2(s, K) f\, ds$

punto sulla curva

Es: Baricentro dell'anello omogeneo



γ definita come $\gamma_1 + \gamma_2$

$\gamma_1(t) = (R\cos t, R\sin t)^T \quad t \in [0, \pi]$

$\gamma_2(t) = (t, 0)^T \quad t \in [-R, R]$

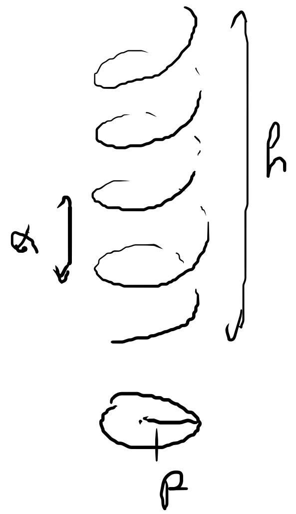
Massa : $\pi R + 2R = R(\pi + 2)$

$\hat{x} = 0$

$$\hat{y} = \frac{1}{M} \int_\gamma y\, ds = \frac{1}{M}\left(\int_{\gamma_1} y\, ds + \underbrace{\int_{\gamma_2} y\, ds}_{=0} \right) = \frac{1}{M} \int_0^\pi R \sin t \cdot \overbrace{\|\gamma_1'(t)\|}^{=R} dt = \frac{R^2}{M} \int_0^\pi \sin t\, dt =$$

$$= \frac{2R^2}{R(\pi+2)} = \frac{2R}{\pi + 2}$$

Es: momento di inerzia risp. asse z dell'elica circolare omogenea di passo α raggio r e altezza h



$$\gamma(t) = (\underbrace{R\cos(2\pi t), R\,\text{sen}(2\pi t)}, \alpha t)^T$$

1 giro $t \in [0,1]$

z si alza di α

$t \in [0, h/\alpha]$

se $t = h/\alpha$ $\quad z = \alpha \cdot \frac{h}{\alpha} = h$

$$I_z = \int_\gamma (x^2+y^2)\, ds = \int_0^{h/\alpha} R^2 \cdot \|\gamma'(t)\|\, dt = \int_0^{h/\alpha} R^2 \cdot \sqrt{4\pi^2R^2+\alpha^2}\, dt =$$

$$= R^2\sqrt{4\pi^2R^2+\alpha^2} \cdot h/\alpha$$

$\gamma'(t) = (-2\pi R\,\text{sen}(2\pi t), 2\pi R\cos(2\pi t), \alpha)^T$

$\|\gamma'(t)\| = \sqrt{4\pi^2R^2+\alpha^2}$

Casi particolari

Sia $g : I \subseteq \mathbb{R} \to \mathbb{R}$ il grafico di g è una curva $\gamma : I \to \mathbb{R}^2$ $\gamma(t) = (t, g(t))^T$

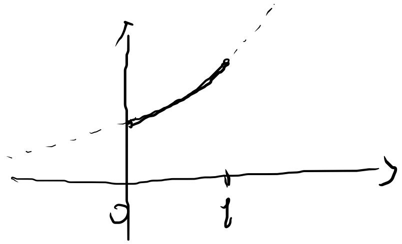
$\gamma(x) = (x, g(x))^T$

$\gamma'(x) = (1, g'(x))^T$

$$\ell(\gamma) = \int_I \|\gamma'(x)\| dx = \int_I \sqrt{1 + g'(x)^2} dx$$

Se $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\Gamma_g \subset A$; $\int_{\Gamma_g} f ds = \int_I f(x, g(x)) \cdot \sqrt{1 + g'(x)^2} dx$

Es: $g : [0,1] \to \mathbb{R}$ $g(x) = e^x$ $g'(x) = e^x$



La lunghezza del grafico di g è $\int_{\Gamma_g} ds = \int_0^1 \sqrt{1 + e^{2x}} dx = *$

$u = \sqrt{1 + e^{2x}}$ $e^{2x} = u^2 - 1$

$du = \frac{1}{\cancel{2}\sqrt{1 + e^{2x}}} \cdot \cancel{2} e^{2x} dx$

$x = 0 \rightsquigarrow u = \sqrt{2}$

$x = 1 \rightsquigarrow u = \sqrt{1 + e^2}$

$$* = \int_{\sqrt{2}}^{\sqrt{1+e^2}} u \cdot \frac{u}{u^2 - 1} du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{2} \frac{1}{u-1} - \frac{1}{2} \frac{1}{u+1}\right) du$$

$$\left[u + \frac{1}{2} \log \frac{u-1}{u+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$\frac{u^2}{u^2-1} = \frac{u^2-1}{u^2-1} + \frac{1}{(u-1)(u+1)} = 1 + \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)$$