Proprietà del determinante

$A \in M_n(\mathbb{K}) \rightsquigarrow \det A \overset{\text{def}}{=} \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)} \in \mathbb{K}$

$A = (a_{ij})$

Matrice diagonale $\quad A = \operatorname{diag}(a_1, \dots, a_n) = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}$

$\Rightarrow \det A = \underbrace{a_1 a_2 \cdots a_n}_{\sigma = id}$

$\sigma \neq id \Rightarrow \exists\, i \in \{1, \dots, n\}$ t.c. $\sigma(i) \neq i$

$a_{i\sigma(i)} = 0$

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{vmatrix} = -12$$

$$\begin{vmatrix} i-1 & 0 \\ 0 & i \end{vmatrix} = i(i-1) = -1-i$$

Matrice triangolare

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \\ & 0 & & \\ 0 & 0 & 0 \cdots 0 & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & & (a_{ij}) \ \ i<j \\ & \ddots & \\ O & & a_{nn} \end{pmatrix}$$ triang. sup.

triangolare superiore

$$\begin{pmatrix} a_{ij} & O \end{pmatrix}$$ $i>j$ triang. inf.

Se A è matrice triangolare (sup. o inf.) allora

$$\det A = a_{11} \cdots a_{nn}$$

$$\begin{pmatrix} a_{11} & & (a_{ij})_{i<j} \\ & \ddots & \\ 0_{(i>j)} & & a_{nn} \end{pmatrix}$$

$$\det A = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, \underline{a_{1\sigma(1)} \cdots a_{n\sigma(n)}}$$

$\sigma = id \rightsquigarrow \underline{a_{11} \cdots a_{nn}}$ (unico termine che rimane) $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$

$\sigma \neq id \Rightarrow \exists\, i \in \{1, \dots, n\}$ t.c. $\underline{\sigma(i) < i} \Rightarrow$

$\underline{a_{i\sigma(i)} = 0}$

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 15 \end{vmatrix} = -30$$

tring sup.

$$\begin{vmatrix} 3 & 0 \\ 5 & 7 \end{vmatrix} = 21$$

tring. inf

Prop. Sia $A \in M_n(\mathbb{K})$ e supponiamo che la i-esima riga sia combinazione lineare di r vettori riga $\in \mathbb{K}^n$:

$$A^{(i)} = \alpha_1 B_1 + \dots + \alpha_r B_r \quad , \quad \begin{cases} \alpha_t \in \mathbb{K} \\ B_t \in \mathbb{K}^n \text{ vettori riga.} \end{cases}$$

Allora: $$\det A = \alpha_1 \begin{vmatrix} A^{(1)} \\ \vdots \\ A^{(i-1)} \\ B_1 \\ \vdots \\ A^{(n)} \end{vmatrix} + \dots + \alpha_r \begin{vmatrix} A^{(1)} \\ \vdots \\ A^{(i-1)} \\ B_r \\ \vdots \\ A^{(n)} \end{vmatrix}$$

OSS A meno di passare alle trasposte si ottiene un risultato analogo sulle colonne:

$$\left| A_{(1)} \cdots A_{(i-1)} \; A_{(i)} \cdots A_{(n)} \right| = \alpha_1 \left| A_{(1)} \cdots B_1 \cdots A_{(n)} \right| + \cdots +$$
$$+ \alpha_r \left| A_{(1)} \cdots B_r \cdots A_{(n)} \right|$$

$$A_{(i)} = \alpha_1 B_1 + \cdots + \alpha_r B_r$$

$\alpha_t \in \mathbb{K}$, $B_t \in \mathbb{K}^n$

vettori colonna

i-esima colonna

$r = 3$

$$A^{(i)} = \alpha_1 B_1 + \underbrace{\alpha_2 B_2 + \alpha_3 B_3} =$$

$$= \underline{\alpha_1 B_1 + C}$$

Es

$$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (1 \;\; 0) + 4\,(0 \;\; 1) \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + 4 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = -3 + 4 \cdot 2 = \underline{\underline{5}}$$

$$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = \underline{\underline{5}}$$

Dim (della proposizione)

Possiamo assumere $r = 2$ (altrimenti il ragionamento si ripete)

Supponiamo $A^{(i)} = \alpha B + \beta C$ $\quad B, C \in \mathbb{K}^n$ vettori riga
$\alpha, \beta \in \mathbb{K}$

Vogliamo far vedere che

$$\det A = \alpha \begin{vmatrix} A^{(1)} \\ \vdots \\ B \\ \vdots \\ A^{(n)} \end{vmatrix} + \beta \begin{vmatrix} A^{(1)} \\ \vdots \\ C \\ \vdots \\ A^{(n)} \end{vmatrix}$$

riga i-esima

$B = (b_1, \dots, b_n)$

$C = (c_1, \dots, c_n)$

$$\boxed{a_{ij} = \alpha b_j + \beta c_j}$$

$$\det A = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots \underbrace{a_{i\sigma(i)}}_{\substack{\| \\ \alpha b_{\sigma(i)} + \beta c_{\sigma(i)}}} \cdots a_{n\sigma(n)} =$$

$$p \cdot (\alpha \cdot b_{\sigma(i)} + \beta c_{\sigma(i)}) = \alpha p b_{\sigma(i)} + \beta p c_{\sigma(i)}$$

$$= \alpha \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots b_{\sigma(i)} \cdots a_{n\sigma(n)} +$$

$$+ \beta \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots c_{\sigma(i)} \cdots a_{n\sigma(n)} = \alpha \begin{vmatrix} A^{(1)} \\ \vdots \\ B \\ \vdots \\ A^{(n)} \end{vmatrix} + \beta \begin{vmatrix} A^{(1)} \\ \vdots \\ C \\ \vdots \\ A^{(n)} \end{vmatrix}$$

Es.

$$\begin{vmatrix} 4 & -2 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{matrix} -2 \\ 6 \end{matrix} \end{vmatrix} =$$

$$= 4 \begin{vmatrix} 1 & -2 \\ 0 & 6 \end{vmatrix} + 5 \begin{vmatrix} 0 & -2 \\ 1 & 6 \end{vmatrix} =$$

$$= 4 \cdot 6 + 5 \cdot 2 = 34$$

$$= 24 + 10 = 34$$

Scambio di 2 righe (o colonne) $\qquad h < k$

$$A \in M_n(\mathbb{K}) \qquad A = \begin{pmatrix} A^{(1)} \\ \vdots \\ A^{(h)} \\ \vdots \\ A^{(k)} \\ \vdots \\ A^{(n)} \end{pmatrix} \rightsquigarrow A' = \begin{pmatrix} A^{(1)} \\ \vdots \\ A^{(k)} \\ \vdots \\ A^{(h)} \\ \vdots \\ A^{(n)} \end{pmatrix}$$

Prop. $\det A' = -\det A$

Dim $A = (a_{ij})$, $A' = (a'_{ij})$

$$a'_{ij} = \begin{cases} a_{ij} & \text{se } i \neq h, k \\ a_{kj} & \text{se } i = h \\ a_{hj} & \text{se } i = k \end{cases}$$

$$a'_{hj} = a_{kj}, \quad a'_{kj} = a_{hj}$$

$h < k$

$$\det A' = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, a'_{1\sigma(1)} \cdots a'_{h\sigma(h)} \cdots a'_{k\sigma(k)} \cdots a'_{n\sigma(n)} =$$

$$a'_{1\sigma(1)} = a_{1\sigma(1)}, \qquad a'_{h\sigma(h)} = a_{k\sigma(h)}, \qquad a'_{k\sigma(k)} = a_{h\sigma(k)}, \qquad a'_{n\sigma(n)} = a_{n\sigma(n)}$$

$$= -\sum_{\sigma\circ\tau \in \Sigma_n} \operatorname{sgn}(\sigma\circ\tau)\, a_{1(\sigma\circ\tau)(1)} \cdots a_{h(\sigma\circ\tau)(h)} \cdots a_{k(\sigma\circ\tau)(k)} \cdots a_{n(\sigma\circ\tau)(n)}$$

Poniamo $\tau = (h \ \ k) \in \Sigma_n$

$\tau(h) = k$

$\tau(k) = h$

$\tau(i) = i \quad \forall\, i \neq h, k$

$\sigma' \in \Sigma_n$

$\sigma' = \sigma' \circ \tau \circ \tau = \sigma \circ \tau$

$\sigma = \sigma' \circ \tau$

$\tau^2 = id$

$$\operatorname{sgn}(\sigma\circ\tau) = \operatorname{sgn}\sigma \cdot \operatorname{sgn}\tau =$$
$$= -\operatorname{sgn}\sigma$$

$\sigma' = \sigma\circ\tau \rightsquigarrow$ $$\det A' = -\sum_{\sigma' \in \Sigma_n} \operatorname{sgn}(\sigma')\, a_{1\sigma'(1)} \cdots a_{n\sigma'(n)} = -\det A$$

A

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{22}a_{11} = -|A|$$

A'

$$\det : \underbrace{\mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{n \text{ volte}} \longrightarrow \mathbb{K}$$

vettori colonne (o alternativamente vettori riga)

$$\left(A_{(1)}, \cdots, A_{(n)} \right) \longrightarrow \left| A_{(1)} \cdots A_{(n)} \right|$$

è lineare in ciascuna colonna

$\boxed{\text{det è multi\_lineare}}$

Scambiando due colonne det cambia segno; det è alternante

Teorema $\det : \underbrace{\mathbb{K}^n \times \dots \times \mathbb{K}^n}_{n \text{ volte}} \longrightarrow \mathbb{K}$ è una

funzione multilineare alternante

$$\det(v_1, \dots, v_n) = |v_1 \dots v_n|$$

$v_i \in \mathbb{K}^n$ vettori colonna

$$\det(v_1, \dots, \alpha v_i, \dots v_n) = \alpha \det(v_1, \dots, v_n)$$

$$\det(v_1, \dots \underbrace{v_i + w_i}, \dots, v_n) = \det(v_1, \dots, v_n) + \det(v_1, \dots, w_i, \dots v_n)$$

$\forall i = 1, \dots, n$

tutto $\times\, \alpha$

$$\det(\alpha v_1, \dots, \alpha v_n) =$$
$$= \alpha^n \det(v_1, \dots, v_n)$$

$$|\alpha A| = \alpha^n |A|$$

Corollario $\det(v_1, \dots, \underset{i}{v_i}, \dots, \underset{j}{v_i}, \dots, v_n) = 0$ se $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Dim. $\det(v_1, \dots, v_i, \dots, v_i, \dots, v_n) = -\det(v_1, \dots, v_i, \dots, v_i, \dots, v_n)$

$\Longrightarrow 2 \det(v_1, \dots, v_n) = 0$

$\Longrightarrow \det(v_1, \dots, v_n) = 0$ se $2 \neq 0$ in $\mathbb{K}$

(non vale in $\mathbb{Z}_2$)