2 Dicembre

Def u è una soluzione debole locale di NS in $(a,b)\times U$ se

1) $u \in L^\infty((a,b), L^2(U))$ e $\nabla u \in L^2((a,b), L^2(U))$
$= L^2((a,b)\times U, \mathbb{R}^9)$

2) $\int_a^b (\langle u, \Delta\phi\rangle + \langle u, \partial_t \phi\rangle - \langle \operatorname{div}(u\otimes u), \phi\rangle)\, dt' = 0$
$\forall\ \phi \in C^\infty_{c0}((a,b)\times U, \mathbb{R}^3)$.

$\nabla\cdot u = 0$

Esempio $u(t,x) = \alpha(t)\nabla\psi(x)$ $\quad \psi : U \to \mathbb{R}$ armonica

$\nabla\cdot u = \alpha(t)\Delta\psi \equiv 0$

$\alpha \in L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$ è una soluzione debole locale.

$\langle \alpha(t)\nabla\psi, \Delta\phi\rangle = \alpha(t)\langle \nabla\Delta\psi, \phi\rangle = 0$

$\alpha(t)\langle \nabla\psi, \partial_t\phi\rangle = \alpha(t)\langle \partial_j\psi, \partial_t\phi^j\rangle = -\alpha(t)\langle \psi, \partial_t \overset{0}{\widetilde{\partial_j\phi^j}}\rangle = 0$

$\alpha^2(t)\langle \operatorname{div}(\nabla\psi\otimes\nabla\psi), \phi\rangle = \alpha^2\langle \partial_k(\partial_j\psi\,\partial_k\psi), \phi_j\rangle =$

$= \alpha^2\langle \partial_j\partial_k\psi\ \partial_k\psi, \phi_j\rangle = \frac{\alpha^2}{2}\langle \partial_j(\partial_k\psi)^2, \phi_j\rangle =$

$= -\sum_k \frac{\alpha^2}{2}\langle (\partial_k\psi)^2, \overset{0}{\widetilde{\operatorname{div}\phi}}\rangle = 0$

Nel teor 11.1 dimostreremo $u \in C^{0,\gamma}_t([t_0,T], C^0_x(\overline{\Omega}))$

$\overline{\Omega} \subset\subset U$

Def Per $(t_0,x_0)\in \mathbb{R}\times\mathbb{R}^3$ e per $R>0$

$$Q^*_R(t_0,x_0) = \left(t_0-\frac{R^2}{2},\, t_0+\frac{R^2}{2}\right)\times B_R(x_0)$$

$$\boxed{Q_R(t_0,x_0) = (t_0-R^2,\, t_0)\times B_R(x_0)}$$

Teor 11.6 Sia u una soluzione debole locale in $Q_R(t_0,x_0)$

Allora, se

$u\in L^{q'}L^q(Q_R(t_0,x_0))$ con $\frac{2}{q'}+\frac{3}{q} < 1$,

allora u è liscia in x in $\overline{Q_{R'}(t_0,x_0)}$ $R'\in(0,R)$

Q_R

$(q',q)\neq(\infty,3)$

Teor 11.7 $\exists\ \epsilon_{q'q} > 0$ t.c. se u è una soluzione debole in $Q_R(t_0, x_0)$ e se

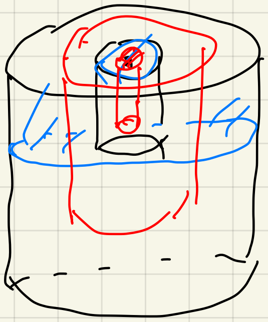
$$\boxed{\| u \|_{L^{q'} L^{q}(Q_R(t_0,x_0))} < \epsilon_{q',q}},$$

$$\frac{2}{q'} + \frac{3}{q} \le 1$$

allora $u \in L^\infty_t H^k_x(\overline{Q_{R'}}(t_0, x_0))$ $\forall\ R' \in (0,R)$ e $\forall\ k \in \mathbb{N}$

$(q', q) = (\infty, 3)$

$q' < \infty$



$Q_\rho(t_0, x_0)$

$q' < \infty$

$$\| u \|_{L^{q'} L^{q}(Q_\rho)} \le \| u \|_{L^{q'} L^{q}((t_0 - \rho^2) \times B(x_0, R))} \xrightarrow{\rho \to 0^+} 0$$

Prop 11.13 $u \in L^{q'} L^{q}(Q_R(t_0, x_0))$ $\quad \frac{2}{q'} + \frac{3}{q'} < 1$

$\Rightarrow u \in L^\infty W^{k,\infty}(\overline{Q_{R'}(t_0, x_0)})$ con $0 < R' < R$

Dim $R' = \frac{R}{2}$

$$\int_{t_0 - R^2}^{t_0} (\langle w, \partial_t \phi \rangle + \langle w, \Delta \phi \rangle + \langle w, u \cdot \nabla \phi \rangle - \langle u, w \cdot \nabla \phi \rangle)\, dt'$$

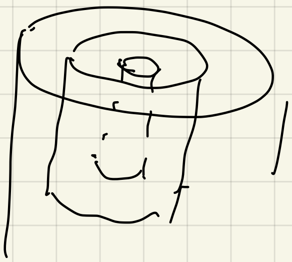
$\forall\ \phi \in C^\infty_c(Q_R(t_0, x_0), \mathbb{R}^3)$

$Q_{R'} \subset Q_R$

$0 < R' < R$

$$\partial_t w - \Delta w = w \cdot \nabla u - u \cdot \nabla w$$

$0 < R'' < R' < R$



$\varphi \in C^\infty_c(\mathbb{R}^4, [0,1])$

$\operatorname{supp} \varphi \cap \overline{Q_R} \subseteq \overline{Q_{R'}}$

$\varphi_{|Q_{R''}} = 1$

$$\partial_t w - \Delta w = w \cdot \nabla u - u \cdot \nabla w \qquad \varphi$$

$$W = \varphi w$$

$$W_j \partial_j u_i - \phi u_j \partial_j w_i$$

$$\partial_t W_i - \Delta W_i = W \cdot \nabla u - \phi\, u \cdot \nabla w + (\varphi_t - \Delta \varphi) w - 2 \nabla \phi \cdot \nabla w \qquad \cancel{\ }$$

$$\varphi(\partial_t w - \Delta w) = (\partial_t - \Delta) W - (\partial_t - \Delta)\varphi\, w + 2 \nabla \varphi \cdot \nabla w$$

$$\partial_t(\varphi w) - \Delta(\varphi w) = \varphi(\partial_t w - \Delta w) + \partial_t \varphi\, w - \Delta \varphi\, w - 2 \nabla \varphi \cdot \nabla w$$

$$-2 \nabla \phi \cdot \nabla w_i = -2 \partial_j (w_i \partial_j \varphi) + 2 w_i \Delta \varphi$$

$$\partial_t W_i - \Delta W_i = \partial_j (W_j u_i - u_j W_i) - 2 \partial_j (w_i \partial_j \varphi)$$

$$+ (\partial_t + \Delta)\varphi\, w_i - \partial_j \varphi\, (w_j u_i - w_i u_j)$$

1) $w \in L^\infty L^\infty (Q_{3R/4})$

2) $1 \Rightarrow \quad w \in L^\infty W^{k,\infty} (\overline{Q}_{R/2}) \qquad \forall k \in \mathbb{N}$

$$\partial_t w - \Delta w = w\cdot\nabla u - u\cdot\nabla w \qquad w \in L^\infty L^\infty(Q_{\frac{3}{4}R})$$

$$w \in L^\infty W^{k,\infty}(Q_{\underbrace{\frac{3}{4}R}_{R_k}}) \qquad k=0$$

$$R'_k \in \left(\frac{R}{2}, R_k\right) \implies u \in L^\infty W^{k,\infty}(Q_{R'_k})$$

Fisso $R''_k \in \left(\frac{R}{2}, R'_k\right)$

$$\varphi \in C^\infty_c(\mathbb{R}^4, [0,1]) \qquad \operatorname{supp}\varphi \cap \overline{Q_{R_k}} \subseteq \overline{Q_{R'_k}}$$

$$\varphi|_{\overline{Q_{R''_k}}} = 1 \qquad W = \varphi w$$

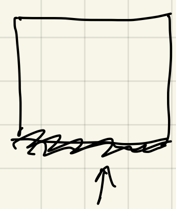
$L^\infty W^{k,\infty}(Q_{R'})$ $\quad$ $L^\infty W^{k,\infty}(Q_{R'})$ $\quad$ $Q_{R'_k}$

$$\partial_t W_i - \Delta W_i = \partial_j(W_j u_i - u_j W_i) - 2\partial_j(w_i \partial_j\varphi) + (\partial_t+\Delta)\varphi\, w_i - \partial_j\varphi\,(w_j u_i - w_i u_j) \qquad (*W)$$

Proposition 11.10. *Assume that $W_t - \triangle W = \partial f$ in $Q_R(t_0,x_0)$ and $W(t_0 - R^2) \equiv 0$. Assume f vanishes outside $\overline{Q}_{\rho_s R}(t_0,x_0)$ for an $\rho_s \in (0,1)$. Then, for any $\rho_i \in (0,\rho_s)$:*

$Q_{R''_k}$ $\quad$ Q_{R_k}

1. *$f \in L^\infty L^\infty(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{0,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*



2. *$f \in L^\infty W^{k,\infty}(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{k,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*
3. *$f \in L^\infty C^{0,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$;*
4. *$f \in L^\infty C^{k,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla^{k+1} W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$.*

Finally, we will use the following regularity result.

Proposition 11.11. *Assume that $W_t - \triangle W = f$ in $Q_R(t_0,x_0)$ and $W(t_0 - R^2) \equiv 0$. Assume f vanishes outside $\overline{Q}_{\rho_s R}(t_0,x_0)$ for an $\rho_s \in (0,1)$. Then, for any $\rho_i \in (0,\rho_s)$:*

1. *$f \in L^\infty L^\infty(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{1,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*
2. *$f \in L^\infty W^{k,\infty}(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{k+1,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*
3. *$f \in L^\infty C^{0,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla^2 W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$;*
4. *$f \in L^\infty C^{k,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla^{k+2} W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$.*

$$W \in L^\infty_t C^{k,\alpha}_x\left(\overline{Q_{R''_k}}\right) \qquad \forall\ \alpha \in (0,1) \qquad W = \varphi w \qquad \varphi|_{\overline{Q_{R''_k}}} = 1$$

$$w \in L^\infty_t C^{k,\alpha}_x\left(\overline{Q_{R''_k}}\right)$$

$$\frac{R}{2} < R'''_k < R''_k \implies u \in L^\infty_t C^{k,\alpha}_x\left(\overline{Q_{R'''_k}}\right)$$

$$\left(\partial_t w - \Delta w = w \cdot \nabla u - u \cdot \nabla w\right)$$

$$R^{(4)}_k \in \left(\frac{R}{2}, R'''_k\right)$$

$$\varphi \in C^\infty_c(\mathbb{R}^4, [0,1]) \qquad \operatorname{supp}\varphi \cap \overline{Q_{R_k}} \subseteq \overline{Q_{R'''_k}}$$

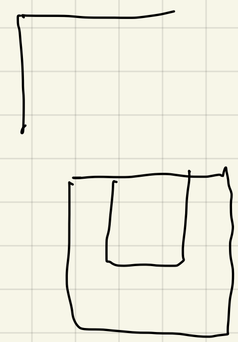
$$\varphi|_{\overline{Q_{R^{(4)}_k}}} = 1 \qquad W = \varphi w$$

$$\partial_t W_i - \Delta W_i = \partial_j (W_j u_i - u_j W_i) - 2\partial_j (w_i \partial_j \varphi) \qquad L^\infty_t C^{k,\alpha}_x(\overline{Q}_{R'''_k})$$
$$+ (\partial_t + \Delta)\varphi\, w_i - \partial_j \varphi\, (w_j u_i - w_i u_j)$$

Proposition 11.10. *Assume that $W_t - \Delta W = \partial f$ in $Q_R(t_0, x_0)$ and $W(t_0 - R^2) \equiv 0$. Assume f vanishes outside $\overline{Q}_{\rho_s R}(t_0, x_0)$ for an $\rho_s \in (0,1)$. Then, for any $\rho_i \in (0, \rho_s)$:*

$Q_{R^{(4)}_k}$

1. *$f \in L^\infty L^\infty(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{0,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*
2. *$f \in L^\infty W^{k,\infty}(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{k,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*
3. *$f \in L^\infty C^{0,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$;*
4. *$f \in L^\infty C^{k,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla^{k+1} W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$.*



Finally, we will use the following regularity result.

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1. *$f \in L^\infty L^\infty(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{1,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*
2. *$f \in L^\infty W^{k,\infty}(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{k+1,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha \in (0,1)$;*
3. *$f \in L^\infty C^{0,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla^2 W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$;*
4. *$f \in L^\infty C^{k,\alpha}(Q_R(t_0,x_0))$ for an $\alpha \in (0,1) \Rightarrow \nabla^{k+2} W \in L^\infty_t L^\infty_x(\overline{Q}_{\rho_i R}(t_0,x_0))$.*

$w \in L^\infty_t C^{k,\alpha}_x(\overline{Q}_{R''_k}) \qquad u \in L^\infty_t C^{k,\alpha}_x(\overline{Q}_{R'''_k})$

$$\Rightarrow \quad W \in L^\infty_t C^{k,\alpha}_x(\overline{Q}_{R^{(4)}_k})$$
$$\nabla^{k+1} W \in L^\infty_t L^\infty_x(\overline{Q}_{R^{(4)}_k})$$
$$W \in L^\infty_t W^{k+1}_x(\overline{Q}_{R^{(4)}_k}) \qquad . \text{ Qui } W = w$$

$$w \in L^\infty_t W^{k+1}_x(\overline{Q}_{R^{(4)}_k}) \qquad R_{k+1} = R^{(4)}_k$$

Per induzione resta definito $\quad R_k \in (\frac{R}{2}, \frac{3}{4}R)$

con $\quad u, w \in L^\infty_t W^{k,\infty}_x(\overline{Q_{R_k}})$

$\Longrightarrow \quad u \in L^\infty_t W^{k,\infty}_x(\overline{Q_{R/2}})$

$$\frac{2}{q'} + \frac{3}{q} < 1$$

$$\boxed{w \in L^\infty L^\infty(Q_{\frac{3R}{4}})}$$

$\frac{3}{4} < \rho_i < \rho_e < 1 \qquad Q_R \qquad Q_{\rho_e R} \qquad Q_{\rho_i R}$

$\phi \in C^\infty_c(\mathbb{R}^4, [0,1]) \qquad \mathrm{supp}\, \phi \cap \overline{Q_R} \subset \overline{Q_{\rho_e R}}$

$\phi|_{Q_{\rho_i R}} = 1. \qquad W = \phi w$

$$\partial_t W_i - \Delta W_i = \partial_j(W_j u_i - u_j W_i) - 2\partial_j(w_i \partial_j \varphi)$$
$$+ (\partial_t + \Delta)\varphi\, w_i - \partial_j \varphi\, (w_j u_i - w_i u_j)$$

$w \in L^{m'} L^m(Q_R) \longleftarrow \qquad w \in L^\infty L^\infty(Q_{\frac{3}{4}R})$

$w \in L^2 L^2(Q_R)$

$\nabla u \in L^2 L^2(Q_R)$