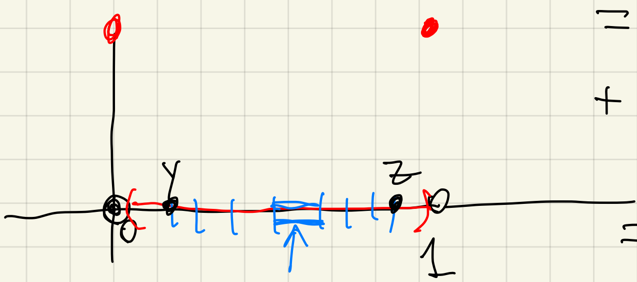
~~Dimostrare~~ Per $f(x)=0$ per $0<x<1$

$f(x)=1$ per $x=0,1$. Dimostrare che e' integrabile per Darboux.

Se f e' integrabile allora $\underline{\int_0^1} f(x) \le \overline{\int_0^1} f(x)$

$s(\Delta)=0 \Rightarrow \underline{\int_0^1} f(x)\,dx = 0$

$$S(\Delta)=\sum_{j=1}^{n}(x_j-x_{j-1})\sup f([x_{j-1},x_j]) =$$



$= (x_1-x_0)\sup f([x_0,x_1])$
$+ (x_n-x_{n-1})\sup f([x_{n-1},x_n])$

$= (x_1-x_0)\sup\{0,1\}$
$+(x_n-x_{n-1})\sup\{0,1\}$

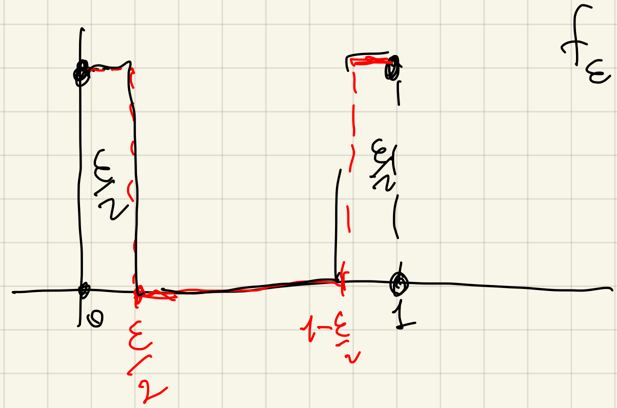
$$S(\Delta)=(\overset{y}{x_1}-x_0)+(x_n-\overset{z}{x_{n-1}})$$

$0\le\overline{\int_0^1} f(x)\,dx \le (x_1-a)+(b-x_{n-1})$ per ogni

$$\boxed{\begin{array}{l} 0\le \overline{\int_0^1} f(x)\,dx \le y-a+b-z \\ \forall \quad a<y\le z<b \end{array}}$$

$< \varepsilon \quad \forall \varepsilon>0$

Per es, $\forall \varepsilon>0$ posso scegliere $0<y-a<\frac{\varepsilon}{2}$
$0<b-z<\frac{\varepsilon}{2}$



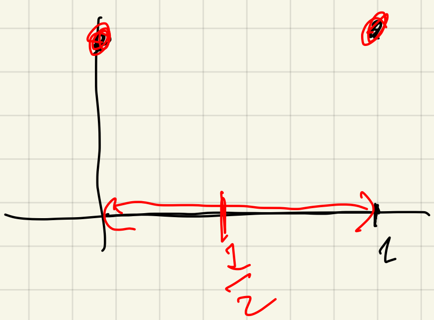
$$f_\varepsilon(x) = \begin{cases} 1 & \text{se } 0 \le x \le \frac{\varepsilon}{2} \\ 1 & 1-\frac{\varepsilon}{2} \le x \le 1 \\ 0 & \frac{\varepsilon}{2} < x < 1-\frac{\varepsilon}{2} \end{cases}$$

$$\int_0^1 f_\varepsilon(x)\,dx = \varepsilon$$

$$0 \le f(x) \le f_\varepsilon(x)$$

$$0 \le \overline{\int_0^1} f(x)\,dx \le \int_0^1 f_\varepsilon(x)\,dx = \varepsilon$$

$$\Longrightarrow \quad 0 \le \overline{\int_0^1} f(x)\,dx \le \varepsilon$$



In $[0,\frac{1}{2}]$ f è decrescente

$\Longrightarrow$ f integrabile

In $[\frac{1}{2},1]$ f è crescente

$\Longrightarrow$ f integrabile

$\Longrightarrow$ f è integrabile in $[0,1]$

$$\underline{\int_0^1} f(x)\,dx = 0 \Longrightarrow \int_0^1 f(x)\,dx = 0$$

Data $f(x) = x \sin\left(\frac{1}{x}\right) e^{-x^2}$ dimostrare che ammette minimo e massimo assoluto ~~sul~~ $\mathbb{R}$.

Ponendo $f(0) = 0$ si ha $f \in C^0(\mathbb{R})$

$$\lim_{x \to \infty} f(x) = 0$$

$\Rightarrow$ $f(\mathbb{R})$ è un insieme limitato

Per Weierstrass esistono punto di max ed un punto di minimo

$g(x) \doteq e^{-x^2}$ $\qquad g(x) > 0 \quad \forall x \in \mathbb{R}$

$$\lim_{x \to +\infty} g(x) = 0$$

$f(x) = x \sin\left(\frac{1}{x}\right) e^{-x^2}$

Siano x_+ t.c. $f(x_+) > 0$

x_- $f(x_-) < 0$

$\varepsilon_0 = \min\{ f(x_+), -f(x_-) \}$

e, siccome

$\forall \varepsilon > 0 \ \exists M_\varepsilon$ t.c. $|x| > M_\varepsilon \Rightarrow |f(x)| < \varepsilon$,

allora considero M_{ε_0}

In $[-M_{\varepsilon_0}, M_{\varepsilon_0}]$ esistono x_m punto di minimo e x_M punto di massimo in $[-M_{\varepsilon_0}, M_{\varepsilon_0}]$

e di conseguenza

$$\boxed{f(x_m) \leq f(x) \leq f(x_M) \quad \forall |x| \leq M_{\varepsilon_0}} \qquad 1$$

Per $|x| > M_{\varepsilon_0}$

$$|f(x)| < \varepsilon_0 \Longleftrightarrow -\varepsilon_0 < f(x) < \varepsilon_0 \qquad 2$$

$\varepsilon_0 = \min\{ f(x_+), -f(x_-) \}$

$\varepsilon_0 \leq f(x_+)$ $\qquad$ $\varepsilon_0 \leq -f(x_-)$

$-\varepsilon_0 \geq f(x_-)$

Per $|x| > M_{\varepsilon_0}$

$f(x)$ $x_\pm \in [-M_{\varepsilon_0}, M_{\varepsilon_0}]$

$f(x_m) \leq f(x_-) \leq -\varepsilon_0 < f(x) < \varepsilon_0$

$\leq f(x_+) \leq f(x_M)$

$o(x^n) + o(x^{n+1}) = o(x^n)$ in 0.

$o(x^{n+1}) = o(x^n)$

$\underbrace{x^{n+1} = o(x^n)}$

$\underbrace{o(x^n) + o(o(x^n))}$

$$\lim_{x\to 0} \frac{o(x^n) + o(x^{n+1})}{x^n} = \lim_{x\to 0}\left[\frac{o(x^n)}{x^n} + \frac{o(x^{n+1})}{x^n}\right]$$

0

$$= \lim_{x\to 0} \frac{o(x^{n+1})}{x^n}\frac{x}{x} = \lim_{x\to 0} \frac{o(x^{n+1})}{x^{n+1}} \cdot x$$

$$= 0 \cdot 0 = 0$$

$$\operatorname{Re}\left(\frac{z^2+2z}{1+z}\right) > 0 \qquad 1$$

$$\operatorname{Re}\left((z^2+2z)\,\frac{1+\bar z}{|1+z|^2}\right) = \frac{1}{|1+z|^2}\,\underline{\underline{\operatorname{Re}\left((z^2+2z)(1+\bar z)\right)}} > 0$$

$$\operatorname{Re}\left((z^2+2z)(1+\bar z)\right) > 0$$

$$\operatorname{Re}\left(z^2+2z+|z|^2 z+2|z|^2\right) > 0$$

$$z = x+iy$$

$$\operatorname{Re}\left(x^2-y^2+\cancel{2ixy}+2x+\cancel{2iy}+(x^2+y^2)x+\cancel{i(x^2+y^2)y}+2x^2+2y^2\right) > 0$$

$$x^2-y^2+2x+x^3+y^2x+2x^2+2y^2 > 0$$

$$3x^2+y^2+x^3+y^2x+2x > 0$$

$$(1+x)\,y^2 > -x^3-3x^2-2x$$

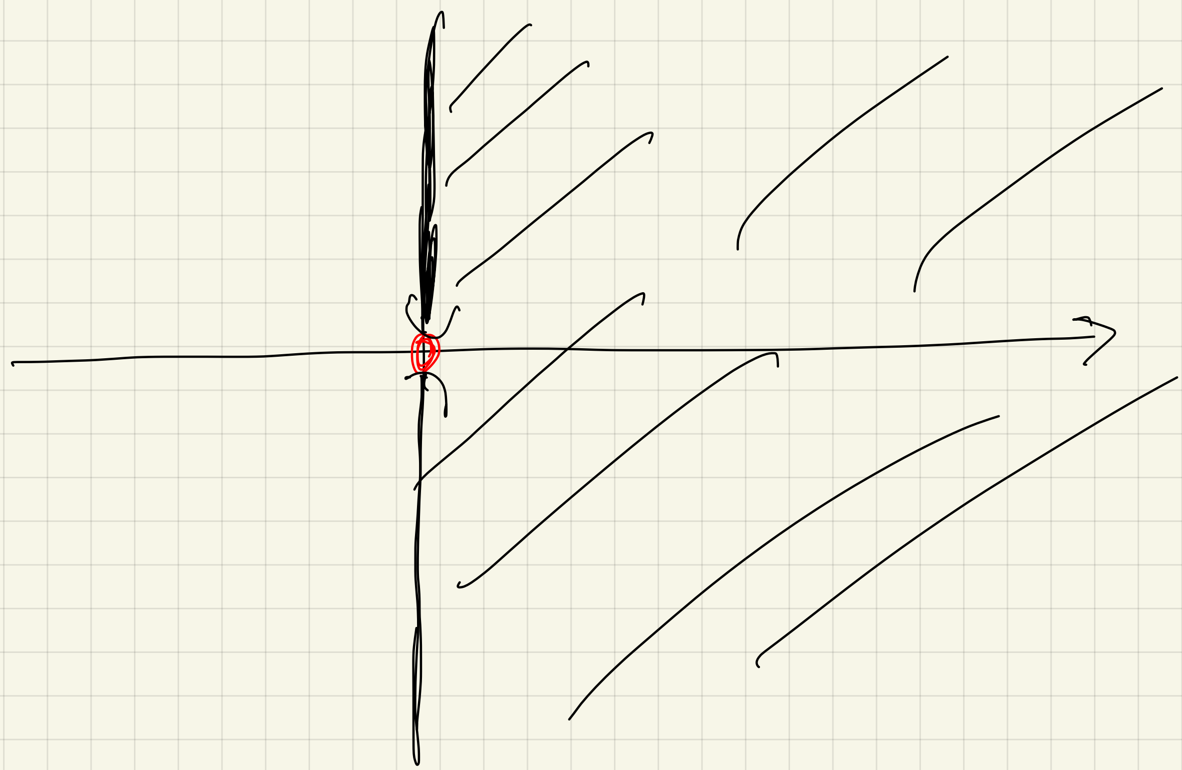
$$(1+x)\,y^2 > -x^3-3x^2-2x$$

Se $1+x>0$, cioè se $x>-1$

$$y^2 > \frac{-x^3-3x^2-2x}{1+x} = f(x)$$

Se $x>0$ ho che $f(x)<0$

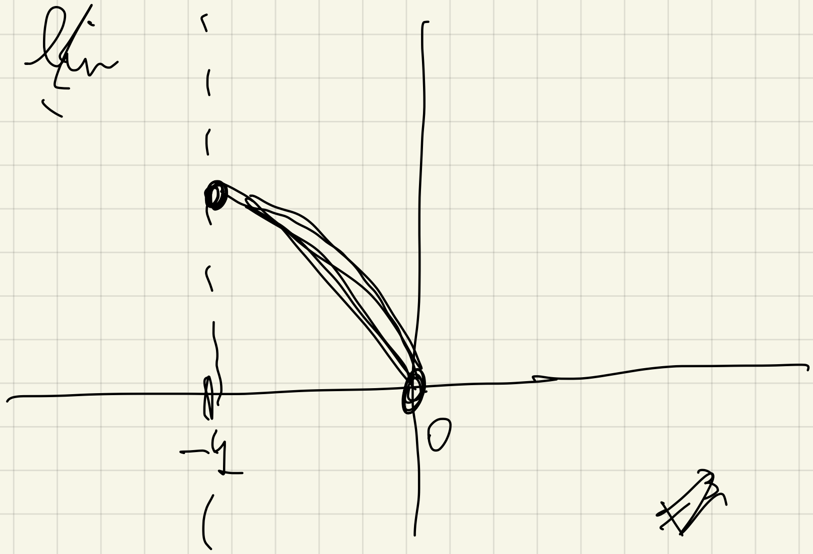
$\Rightarrow \quad y^2 \geq 0 > f(x) \quad \forall\, y \in \mathbb{R}$



Se $x=0$, $f(0)=0$ $\qquad y^2 > f(0) \iff y \neq 0$

$-1<x<0$ $\qquad y^2 > f(x) := \dfrac{-x^3-3x^2-2x}{1+x}$

lim

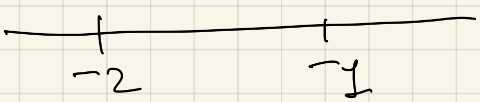


$$\lim_{x \to -1^+}$$

$$(-x)(x^2+3x+2)=0$$

$$x^2+3x+2=0 \qquad x_\pm = -\frac{3}{2} \pm \frac{\sqrt{9-8}}{2}$$

$$= -\frac{3}{2} \pm \frac{1}{2} = \begin{cases} -2 \\ -1 \end{cases}$$



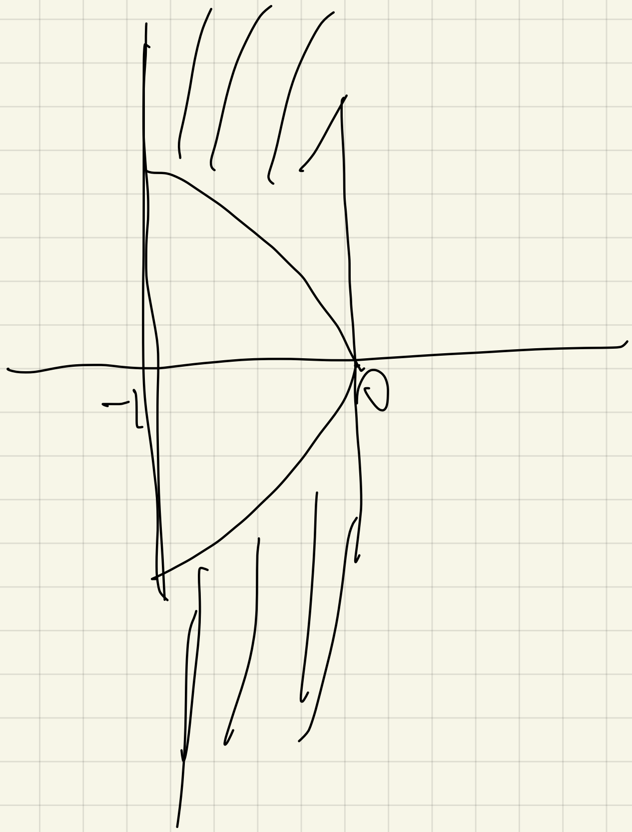
$$\lim_{x\to -1} \frac{-x^3-3x^2-2x}{1+x} = \lim_{x \to -1} \frac{-3x^2-6x-2}{1} = \frac{-3+6-2}{1} = 1$$

$$y^2 > f(x) \qquad -1<x<0$$

$$y > \sqrt{f(x)}$$

$$y < -\sqrt{f(x)}$$

$$\sqrt{f(x)} = \sqrt{\frac{-x^3-3x^2-2x}{1+x}}$$



$1+x<0$

$(1+x)\, y^2 > -x^3-3x^2-2x$ $\qquad x = -1$

$0 > 0$

$$z^6 + z^3 + |z|^2 + 1 = 0$$

$$z = r\cos\vartheta + ir\sin\vartheta = r(\cos\vartheta + i\sin\vartheta)$$

$$r^6\cos(6\vartheta) + ir^6\sin(6\vartheta) + r^3\cos(3\vartheta) + ir^3\sin(3\vartheta)$$
$$+ r^2 + 1 = 0$$

$$\begin{cases} r^6\cos(6\vartheta) + r^3\cos(3\vartheta) + r^2 + 1 = 0 \\ r^3\left(r^3\sin(6\vartheta) + \sin(3\vartheta)\right) = 0 \end{cases}$$

$$\sin(6\vartheta) = 2\sin(3\vartheta)\cos(3\vartheta)$$

$$r^3 \sin(3\vartheta)\left(2r^3\cos(3\vartheta) + 1\right) = 0$$

$r = 0$ non risolve il sistema

$$\sin(3\vartheta) = 0 \Rightarrow \cos(3\vartheta) = \begin{cases} 1 \\ -1 \end{cases}$$

$$\cos(6\vartheta) = 2\cos^2(3\vartheta) - 1 = 1$$

per $\cos(3\vartheta) = 1$

$r^6 + r^3 + r^2 + 1 = 0$ non ha soluzioni

$\cos(3\vartheta) = -1$

$r^6 - r^3 + r^2 + 1 = 0$ non ha soluzioni

$$2r^3\cos(3\vartheta)+1=0$$

$$r^6\cos(6\vartheta)+r^3\cos(3\vartheta)+r^2+1=0$$

$$\cos(3\vartheta)=-\frac{1}{2r^3}$$

$$\cos(6\vartheta)=2\cos^2(3\vartheta)-1=2\,\frac{1}{4r^6}-1=$$

$$\boxed{\cos(6\vartheta)=\frac{1}{2r^6}-1}$$

$$r^6\left(\frac{1}{2r^6}-1\right)+r^3\left(-\frac{1}{2r^3}\right)+r^2+1=0$$

$$\cancel{\frac{1}{2}}-r^6-\cancel{\frac{1}{2}}+r^2+1=0$$

$$f(r)=r^6-r^2-1=0 \qquad \boxed{w^3-w-1=0}$$

$$r^2=w$$

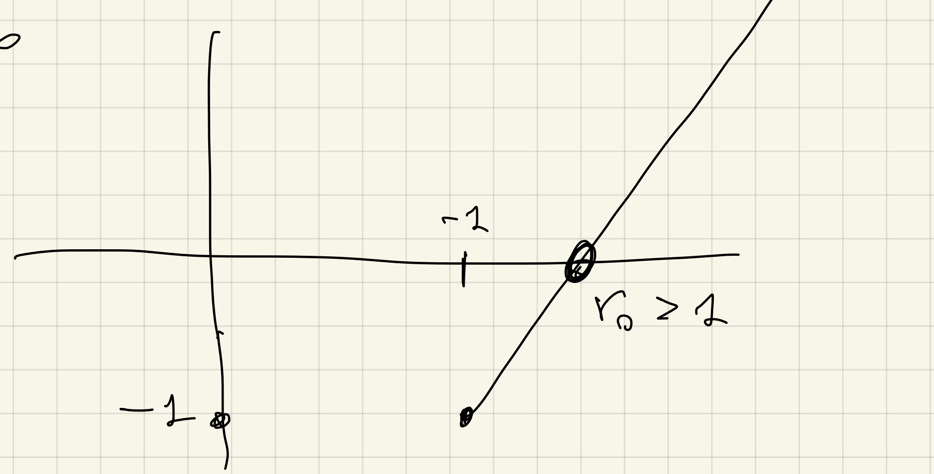
$f(r)<0$ per $0\le r\le 1$ $\qquad r=\sqrt{w}$

$$\lim_{r\to+\infty} f(r)=+\infty$$

$$f'(r)=6r^5-2r=$$

$$=2r(3r^4-1)>0$$

per $r\ge 1$

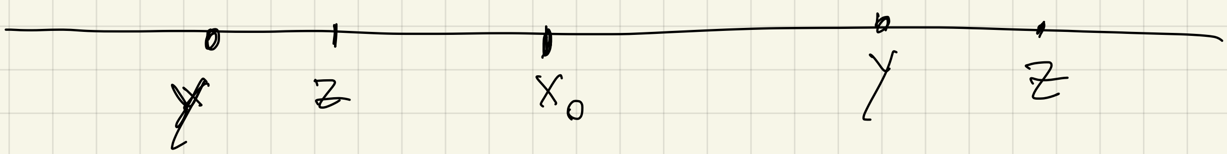


$$\cos(3\vartheta) = -\frac{1}{2r_0} \in \left(-\frac{1}{2}, 0\right)$$

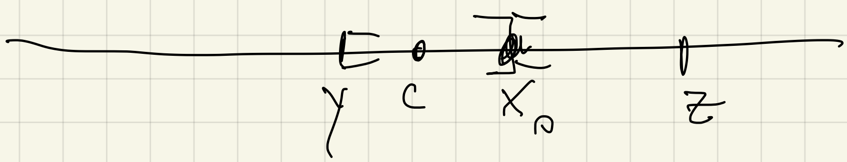
$f : \mathbb{R} \longrightarrow \mathbb{R}$, $\quad X$ t.c. $\#X < +\infty$

$f'(x)$ esiste $\forall x \in \mathbb{R} \setminus X$ con

$f'(x) < 0 \quad \forall x \notin X$



$f(y) > f(z)$



$f(y) > f(x_0) > f(z)$

In $[y, x_0]$ f è continua.

In (y, x_0) f è derivabile. Applico Lagrange:

$\exists \; c \in (y, x_0)$ t.c.

$$0 > f'(c) = \frac{\overbrace{f(y) - f(x_0)}^{>0}}{\underbrace{y - x_0}_{<0}}$$

$f(y) - f(x_0) > 0$

$f(y) > f(x_0)$

Siano f e $g : [a,b] \to \mathbb{R}$ uguali ovunque eccetto in $x_0 \in [a,b]$. Dimostrare che

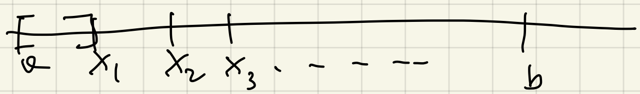
$$\underline{\int_a^b} f(x)\,dx = \underline{\int_a^b} g(x)\,dx$$

Sia $f(x_0) < g(x_0)$ ed inoltre $f(x) = g(x)$ $\forall\, x \neq x_0$

$$\underline{\int_a^b} f(x)\,dx \leq \underline{\int_a^b} g(x)\,dx$$

Supponiamo $x_0 = a$

$$\Delta : \quad x_0 = a < x_1 < x_2 < \dots < x_n = b$$



$$s_f(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf f([x_{j-1}, x_j])$$

$$s_g(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf g([x_{j-1}, x_j])$$

$$0 \leq s_g(\Delta) - s_f(\Delta) = (x_1 - a)\left(\inf g([a, x_1]) - \inf f([a, x_1])\right)$$

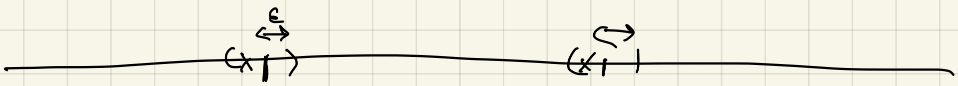
$$< (x_1 - a)\left(\underbrace{\sup g([a,b]) - \inf f([a,b])}_{L}\right)$$

$$0 \leq s_g(\Delta) - s_f(\Delta) < (x_1 - a)\, L$$

Sia per assurdo $\underline{\int_a^b} g(x)\,dx > \underline{\int_a^b} f(x)\,dx$

$$0 \le s_g(\Delta) - s_f(\Delta) < (x_1 - a) L$$

Să presupunem $\underline{\int_a^b} g(x)\,dx > \underline{\int_a^b} f(x)\,dx$



$\underline{\int_a^b} f(x)\,dx$ $\qquad$ $\underline{\int_b^a} g(x)\,dx$

$$\exists\, \Delta_1 \ \text{t.c.} \ \underline{\int_a^b} f \ge s_f(\Delta_1) > \underline{\int_a^b} f - \varepsilon \qquad (*)$$

$$\exists\, \Delta_2 \ \text{t.c.} \ \underline{\int_a^b} g \ge s_g(\Delta_2) \ge \underline{\int_a^b} g - \varepsilon \qquad (**)$$

$$\Delta \preceq \Delta_1, \quad \Delta \preceq \Delta_2$$

$$\underline{\int_a^b} f \ge s_f(\Delta) > \underline{\int_a^b} f - \varepsilon$$

$$\underline{\int_a^b} g \ge s_g(\Delta) \ge \underline{\int_a^b} g - \varepsilon$$

$$s_g(\Delta) - s_f(\Delta) \ge \underline{\int_a^b} g - \varepsilon - \underline{\int_a^b} f =$$

$$= \underline{\int_a^b} g - \underline{\int_a^b} f - \varepsilon \qquad \varepsilon = \frac{1}{10}\left(\underline{\int_a^b} g - \underline{\int_a^b} f\right)$$

$$= \frac{9}{10}\left(\underline{\int_a^b} g - \underline{\int_a^b} f\right)$$

$$\frac{9}{10}\left(\underline{\int_a^b} g - \underline{\int_a^b} f\right) \le s_g(\Delta) - s_f(\Delta) < (x_1 - a) L < \frac{1}{10}\left(\underline{\int_a^b} g - \underline{\int_a^b} f\right)$$

$$\Delta : x_0 < x_1 < \dots < x_n = b$$

$$(x_1 - a) L < \frac{1}{10}\left(\underline{\int_a^b} g - \underline{\int_a^b} f\right)$$