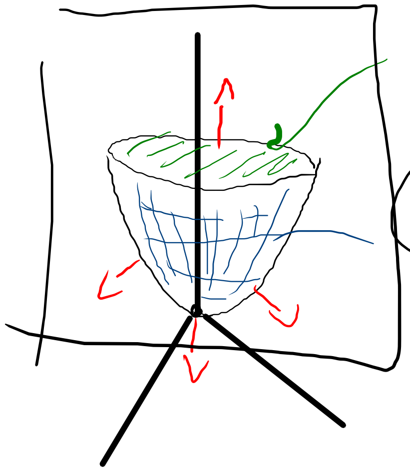


$\sigma_2(s,t) = (s, t, 3)^T \qquad \sigma_2 : B((0,0)^T, \sqrt{3}) \to \mathbb{R}^3 \qquad J\sigma_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^T \qquad \nu = (0,0,1)^T$

$\sigma_1(s,t) = (t, s, s^2+t^2)^T \qquad \sigma_1 : B((0,0)^T, \sqrt{3}) \to \mathbb{R}^3$

$$\Omega \qquad \partial\Omega = \sigma_1 + \sigma_2 \qquad J\sigma_1^T = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{pmatrix} 0 & 1 & 2s \\ 1 & 0 & 2t \end{pmatrix} \end{array} \qquad \nu = (2t, 2s, -1)^T$$

Flusso uscente da Ω del campo $g(x,y,z) = (x, y, z)^T$

$$\text{Flusso} = \iint_{\substack{\sigma \\ \| \\ \partial\Omega^+}} \langle g, \nu \rangle \, d\sigma = \iint_{\sigma_1} \langle g, \nu \rangle \, d\sigma + \iint_{\sigma_2} \langle g, \nu \rangle \, d\sigma = \iint_{B((0,0)^T, \sqrt{3})} \left\langle (t, s, s^2+t^2)^T, (2t, 2s, -1)^T \right\rangle ds\,dt$$

$$+ \iint_{B((0,0)^T, \sqrt{3})} \left\langle (s, t, 3)^T, (0,0,1)^T \right\rangle ds\,dt = \iint_{B((0,0)^T, \sqrt{3})} (\underbrace{2t^2 + 2s^2 - s^2 - t^2}_{t^2+s^2}) \, ds\,dt + \iint_{B((0,0)^T, \sqrt{3})} 3 \, ds\,dt =$$

$$= \int_0^{2\pi} \left(\int_0^{\sqrt{3}} \underbrace{\rho^2 \cdot \rho}_{\rho^3} \, d\rho \right) d\vartheta + 3\pi 3 = \cancel{2}\pi \cdot \frac{1}{\cancel{4}_2} \sqrt{3}^4 + 9\pi = 9\pi \left(\frac{1}{2} + 1 \right) = \frac{27}{2}\pi$$

Operatori differenziali $\quad A \subseteq \mathbb{R}^n$ aperto

gradiente $\quad \nabla : C^1(A, \mathbb{R}) \to C^0(A, \mathbb{R}^n)$ $\qquad \nabla \quad f \rightsquigarrow \nabla f$

campo scalare $\rightsquigarrow$ campo vettoriale

rotore $n=3$ $\quad \mathrm{rot} : C^1(A, \mathbb{R}^n) \to C^0(A, \mathbb{R}^n)$ $\qquad g \rightsquigarrow \mathrm{rot}\, g$

campo vettoriale $\rightsquigarrow$ campo vettoriale

divergenza $\quad \mathrm{div} : C^1(A, \mathbb{R}^n) \to C^0(A, \mathbb{R})$ $\qquad g \xrightarrow{\mathrm{div}} \mathrm{div}\, g$

campo vettore $\xrightarrow{\mathrm{div}}$ campo scalare

se $g(x_1, x_2, \dots, x_n) = \left(X_1(x_1, \dots, x_n), X_2(x_1, \dots, x_n), \dots, X_n(x_1, \dots, x_n)\right)^T$

$$\mathrm{div}\, g = \sum_{k=1}^{n} \frac{\partial X_k}{\partial x_k}$$

è la traccia della Jacobiana

$$\begin{pmatrix} \cdot & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{pmatrix}$$

oss: $\operatorname{rot}(\nabla f) = \bar{0}$ Es. 12

$\operatorname{rot}(\operatorname{div} g)$ NON SI PUÒ FARE!

$$\operatorname{div}(\operatorname{rot} g) = \operatorname{div}\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)^T$$

$$= \frac{\partial^2 Z}{\partial x \partial y} - \frac{\partial^2 Y}{\partial x \partial z} + \frac{\partial^2 X}{\partial y \partial z} - \frac{\partial^2 Z}{\partial y \partial x} + \frac{\partial^2 Y}{\partial z \partial x} - \frac{\partial^2 X}{\partial z \partial y} = 0$$

$$\operatorname{div}(\nabla f) = \Delta f = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}$$ la traccia dell'Hessiana

Laplaciano di f

divergenza

Teorema della divergenza (Gauss)

Sia $\Omega \subset \mathbb{R}^n$ regolare a tratti con bordo orientato positivamente (secondo la normale uscente) $\partial\Omega^+$.

Sia $\overline{\Omega} \subseteq A$ $\quad$ A aperto $\quad$ $g : A \to \mathbb{R}^n$ $\quad$ $g \in C^1(A, \mathbb{R}^n)$.

Allora
$$\int_{\partial\Omega^+} \langle g, \nu \rangle \, d\sigma = \int_\Omega \operatorname{div} g \, dm$$

integrale di superficie ($\mathbb{R}^3$) o di linea ($\mathbb{R}^2$)

integrale di Riemann in $\mathbb{R}^n$

$n = 3$
$$\iint_{\partial\Omega^+ = \sigma} \langle g, \nu \rangle \, d\sigma = \iiint_\Omega \operatorname{div} g(x,y,z) \, dx\,dy\,dz$$

$$\iiint \widetilde{Dg} = \iint \bar{g}$$

ciccione $\quad$ magra

$n = 2$
$$\int_{\partial\Omega^+ = \gamma} \langle g, \nu \rangle \, ds = \iint_\Omega \operatorname{div} g(x,y) \, dx\,dy$$

Df

$$\int_{[a,b]} f'(x)\,dx = \Big[\overset{f}{f(x)} \Big]_a^b \qquad f(b) - f(a)$$

l cucione

mega

0-dimensionale

$$\Omega = \{ (x,y,z)^T : \quad x^2+y^2 \le 3, \quad x^2+y^2 \le z \le 3 \}$$

Flusso uscente da Ω $\qquad g(x,y,z) = (x,y,z)^T$

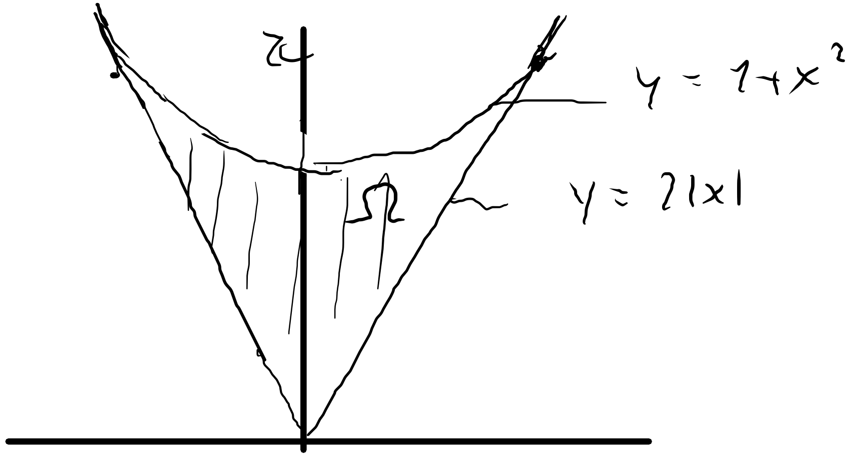
$$\iint_{\partial\Omega^+} \langle g, \nu \rangle \, d\sigma \underset{\text{Gauss}}{=} \iiint_\Omega \operatorname{div} g \, dx\,dy\,dz$$

$$\operatorname{div} g = 1 + 1 + 1 = 3$$

$$\iint = \iiint_\Omega 3 \, dx\,dy\,dz = 3 \int_0^3 \Big(\underbrace{\iint_{S_z} dx\,dy}_{= \pi z} \Big) dz = 3 \cdot \int_0^3 \pi z \, dz = 3\pi \cdot \frac{1}{2} 9 = \frac{27\pi}{2}$$

$x^2+y^2 = z$

$S_z = \{ (x,y)^T \in \mathbb{R}^2 : x^2+y^2 \le z \}$ $\qquad$ Area $= \pi z$



$$\Omega = \{ (x,y)^T \in \mathbb{R}^2 : \ |x| < 1 \ , \ 2|x| < y < 1+x^2 \}$$

$$g(x,y) = (y^2 \operatorname{sen} y, \ x^2 y)^T \qquad \operatorname{div} g = 0 + x^2$$

$$\text{Flusso} = \int_{\partial\Omega^+} \langle g, \nu \rangle \, d\sigma \underset{\text{Gauss}}{=} \iint_\Omega \operatorname{div} g \, dx\,dy =$$

$$\iint_\Omega x^2 \, dx\,dy = \int_{-1}^{1} \left(\int_{2|x|}^{1+x^2} x^2 \, dy \right) dx = \int_{-1}^{1} \left(x^2 + x^4 - 2|x| x^2 \right) dx = 2 \int_0^1 \left(x^2 + x^4 - 2x^3 \right) dx$$

$$= 2 \left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2} \right) = \frac{1}{15}$$

$$\frac{10 + 6 - 15}{30} = \frac{1}{30}$$

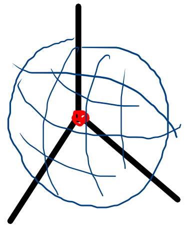
Teorema di Gauss per il campo elettrico

Campo elettrico generato da una carica q

$$E_q = \underbrace{\frac{1}{4\pi\varepsilon_0}\, q\, \frac{(x,y,z)^T}{\|(x,y,z)^T\|^3}}$$

Flusso del campo attraverso una sfera

$B(0,R)$



$$\iint_{\text{sfera} = \partial B(0,R)^+} \langle E_q, \omega \rangle \, d\sigma \overset{?}{=} \cancel{\underbrace{\iiint_{B(0,R)} \operatorname{div} E_q \, dx\,dy\,dz}}$$

NON HA SENSO!

$$\sigma(\varphi,\vartheta) = (R \operatorname{sen}\varphi \cos\vartheta,\ R \operatorname{sen}\varphi \operatorname{sen}\vartheta,\ R\cos\varphi)^T$$

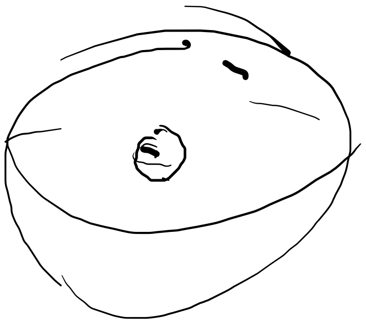
$$\omega(\varphi,\vartheta) = R \operatorname{sen}\varphi \cdot \sigma(\varphi,\vartheta)$$

$$\text{Flusso} = \int_0^{2\pi} \left(\int_0^{\pi} \frac{1}{4\pi\varepsilon_0}\, q \cdot \frac{1}{R^3} \langle \sigma(\varphi,\vartheta), R \operatorname{sen}\varphi\, \sigma(\varphi,\vartheta) \rangle \, d\varphi \right) d\vartheta$$

$R^2 \operatorname{sen}\varphi$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{\cancel{R^2}} \int_0^{2\pi} \left(\underbrace{\int_0^{\pi} \cancel{R^2} \operatorname{sen}\varphi \, d\varphi}_{2} \right) d\vartheta = \frac{q\,4\pi}{4\pi\varepsilon_0} = \frac{q}{\varepsilon_0}$$

Flusso del campo attraverso una sfera di raggio R è



$$\boxed{\operatorname{div} E_q = K \cdot 0}$$

$$\frac{q}{\varepsilon_0}$$

$$K$$

$$\frac{1}{4\pi\varepsilon_0} q$$

$$\frac{(x,y,z)}{\|(\ )\|^3}$$

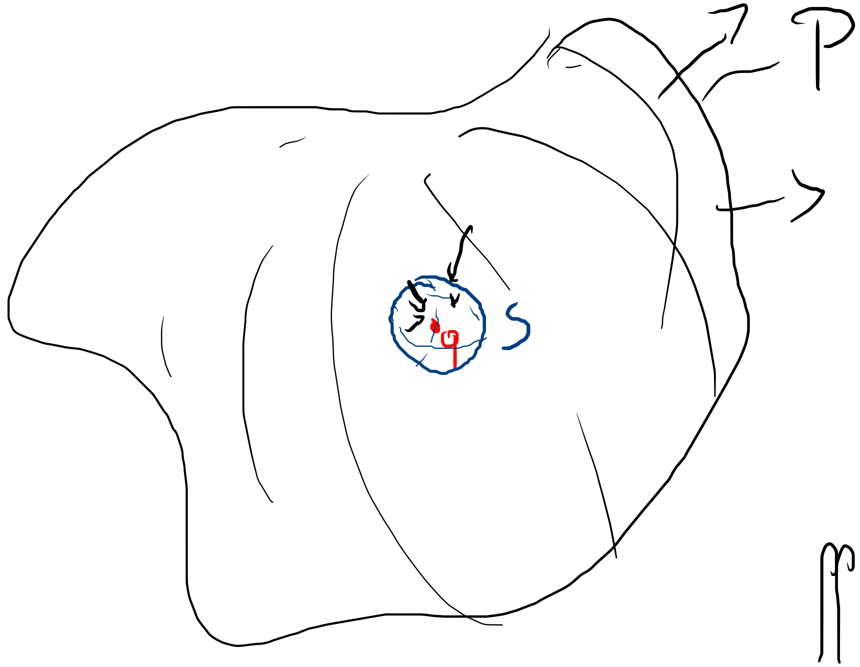
$$\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(\ )} + \frac{\partial}{\partial z}\frac{z}{(\ )}$$

$$\to \frac{(x^2+y^2+z^2)^{3/2} - x\,\frac{3}{\cancel{2}}(x^2+y^2+z)^{1/2}\cdot\cancel{2}x}{(x^2+y^2+z^2)^3} =$$

$$= \frac{\sqrt{x^2+y^2+z^2}\,(x^2+y^2+z^2-3x^2)}{(x^2+y^2+z^2)^3}$$

$$\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{x^2+z^2-2y^2}{(\ )^{5/2}} + \frac{x^2+y^2-2z^2}{(\ )^{5/2}}$$

$$=$$

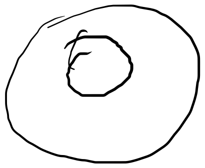


Nella zona P − S il campo E_q è definito e ha divergenza nulla.

Per il Teorema della divergenza di Gauss si ha che il flusso attraverso il bordo di P − S è nullo

$$\iint_{\partial(P-S)^+} \langle E_q , \nu \rangle \, d\sigma = \iiint_{P-S} \operatorname{div} E_q \, dm = 0$$

$$\iint_{\partial P} \langle E_q , \nu \rangle \, d\sigma = \iint_{\partial S^+} \langle E_q , \nu \rangle \, d\sigma = \frac{q}{\varepsilon_0}$$



$$\int_{\partial\Omega^+} \langle g, \nu \rangle \, d\sigma = \iint_\Omega \operatorname{div} g \, dx\,dy$$

Dimostrazione $\underline{N=2}$

Supponiamo che Ω sia un dominio normale rispetto ad entrambi gli assi

$$g(x,y) = \left(X(x,y),\ Y(x,y) \right)^T$$

$$\partial\Omega^+ = \gamma \qquad \gamma(t) = (x(t), y(t))^T \quad t \in I$$

$$\nu(t) = (y'(t), -x'(t))^T$$

Dimostriamo che

$$\int_I \left[X(x(t),y(t)) \cdot y'(t) - Y(x(t),y(t)) \cdot x'(t) \right] dt = \iint_\Omega \left[\frac{\partial X}{\partial x}(x,y) + \frac{\partial Y}{\partial y}(x,y) \right] dx\,dy$$

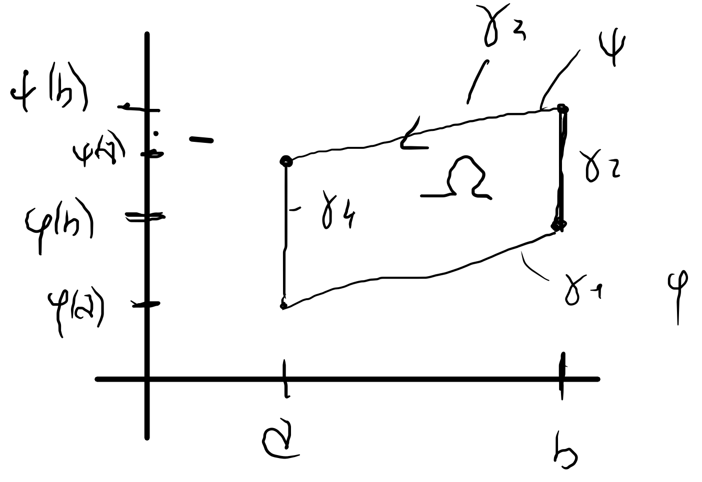
Proveremo che

(1) $$\int_I X y' \, dt = \iint_\Omega \frac{\partial X}{\partial x} \, dx\,dy$$

(2) $$-\int_I Y x' \, dt = \iint_\Omega \frac{\partial Y}{\partial y} \, dx\,dy$$

$$\int_I Y(x(t),y(t)) \cdot x'(t) \, dt =$$

$\gamma : I \to \mathbb{R}^2$ $\quad \left[\Omega = \{ (x,y)^T \in \mathbb{R}^2 ;\ x \in [a,b],\ \varphi(x) \le y \le \psi(x) \} \right]$



$$\gamma_1 : [a,b] \to \mathbb{R}^2 \qquad \gamma_1(t) = (t, \varphi(t))^T \qquad \gamma_1'(t) = (1, \varphi'(t))^T$$

$$\gamma_2 : [\varphi(b), \psi(b)] \to \mathbb{R}^2 \qquad \gamma_2(t) = (b, t) \qquad \gamma_2'(t) = \underline{\underline{(0,1)^T}}$$

$$-\gamma_3 : [a,b] \to \mathbb{R}^2 \qquad -\gamma_3(t) = (t, \psi(t))^T \qquad -\gamma_3'(t) = (1, \psi'(t))^T$$

$$-\gamma_4 : [\varphi(a), \psi(a)] \to \mathbb{R}^2 \qquad -\gamma_4(t) = (a, t)^T \qquad -\gamma_4'(t) = (0,1)^T$$

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

$$\int_\gamma Y(x(t), y(t)) \cdot x'(t)\, dt = \int_a^b Y(t, \varphi(t)) \cdot 1\, dt + \cancel{\int_{\varphi(b)}^{\psi(b)} Y(b,t) \cdot 0\, dt} - \int_a^b Y(t, \psi(t)) \cdot 1\, dt - \cancel{\int_{\varphi(a)}^{\psi(a)} Y(a,t) \cdot 0\, dt}$$

$$= \int_a^b \left[Y(t, \varphi(t)) - Y(t, \psi(t)) \right] dt = \int_a^b \left[Y(x, \varphi(x)) - Y(x, \psi(x)) \right] dx$$

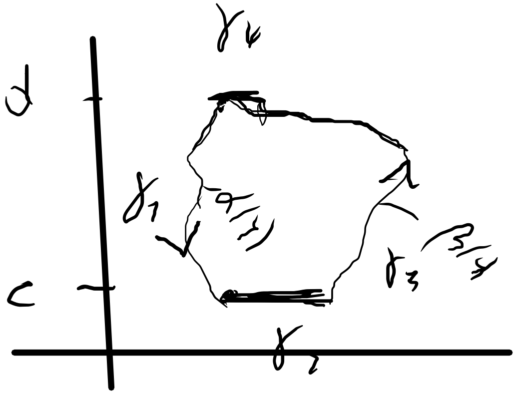
$t \to x$

$$\iint_\Omega \frac{\partial Y}{\partial y}\, dx\, dy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \frac{\partial Y}{\partial y}(x,y)\, dy \right) dx = \int_a^b \left[Y(x, \psi(x)) - Y(x, \varphi(x)) \right] dx$$

Ho dimostrato che $\int_\gamma Y x'\, dt = - \iint_\Omega \frac{\partial Y}{\partial y}\, dx\, dy$

Per dimostrare (1) $\int_\gamma X y' \, dt = \iint_\Omega \frac{\partial X}{\partial x} \, dx\,dy$ scriviamo Ω come insieme normale rispetto all'asse y

$$\Omega = \{ (x,y)^T \in \mathbb{R}^2 : \quad y \in [c,d], \quad \alpha(y) \le x \le \beta(y) \}$$



$$\int_\gamma X \underline{y'} \, dt = -\int_c^d X(\alpha(t),t)\cdot 1 \, dt + \underset{\gamma_2}{0} + \int_c^d X(\beta(t),t)\cdot 1 \, dt + \underset{\gamma_4}{0}$$

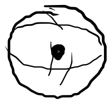
$$= \int_c^d \left[X(\beta(t),t) - X(\alpha(t),t) \right] dt$$

$\gamma_1(t) = (\alpha(t), t)^T \qquad \gamma_1'(t) = (\alpha'(t), 1)^T$

$$= \int_c^d \left(\int_{\alpha(t)}^{\beta(t)} \frac{\partial X}{\partial x}(x,y) \, dx \right) dy = \iint_\Omega \frac{\partial X}{\partial x} \, dx\,dy$$

Significato fisico della divergenza

$(x_0, y_0, z_0)^T \qquad \mathrm{div}\, g\,(x_0, y_0, z_0) \;?$



$B\left((x_0, y_0, z_0)^T, R\right) = B_R$

$$\iiint_{B_R} \mathrm{div}\, g \, dm = \iint_{S_R} \langle g, \nu \rangle \, d\sigma$$

$S_R = \partial B_R$

$$\lim_{R \to 0^+} \underbrace{\frac{1}{\mathrm{Vol}(B_R)} \iiint_{B_R} \mathrm{div}\, g(x,y,z)\, dx\, dy\, dz}_{= \,\mathrm{div}\, g(\underbrace{x_R, y_R, z_R}_{\in B_R})} = \mathrm{div}\, g(x_0, y_0, z_0)$$

$$\mathrm{div}\, g(x,y,z) = \lim_{R \to 0^+} \frac{\iint_{S_R} \langle g, \nu \rangle \, d\sigma}{\mathrm{Vol}(B_R)}$$

Se $\mathrm{div}\, g(x,y,z) = 0$ il flusso è nullo

Un campo g si dice solenoidale se esiste un potenziale vettore $h: A \to \mathbb{R}^3$ tale che $\operatorname{rot} h = g$

$$\begin{cases} g \text{ conservativo} \Rightarrow g = \nabla U \quad U \text{ campo scalare} \Rightarrow \operatorname{rot} g = \operatorname{rot} \nabla U = 0 \\ g \text{ solenoidale} \Rightarrow g = \operatorname{rot} h \quad h \text{ campo vettoriale} \Rightarrow \operatorname{div} g = \operatorname{div} \operatorname{rot} h = 0 \end{cases}$$

g conservativo $\Rightarrow$ $\oint_\gamma \langle g, \tau \rangle \, ds = 0$ circuitazione nulla

solenoidale $\oiint_S \langle g, \nu \rangle \, d\sigma = 0$

superficie chiusa