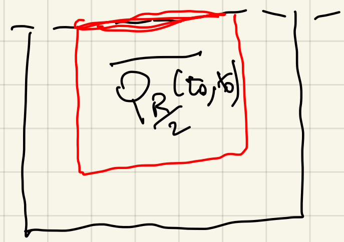
7 dicembre

Prop 11.12 u soluz. debole locale in $Q_R(t_0,x_0)$

e se $u \in L^{q'}L^{q}(Q_R(t_0,x_0))$ $\frac{2}{q'}+\frac{3}{q}<1$

allora u è liscio in x in $\overline{Q_{R'}(t_0,x_0)}$ $0<R'<R$



Abbiamo ottenuto $\omega \in L^\infty L^\infty(Q_{\frac{3}{4}R}(t_0,x_0))$ ~~$\forall$~~

Dimostriamo che $\not\exists$.

Si parte $\omega \in L^{m'}L^{m}(Q_R(t_0,x_0))$ (1)

$\nabla u \in L^2L^2(Q_R(t_0,x_0)) \Rightarrow$ (1) almeno per $(m',m)=(2,2)$

$$\partial_t \omega - \Delta \omega = (\omega\cdot\nabla)u - (u\cdot\nabla)\omega$$

$\frac{3}{4} < \rho_i < \rho_e < 1$

$\phi \in C_c^\infty(\mathbb{R}^4,[0,1])$

$\mathrm{supp}\,\phi \cap \overline{Q_R(t_0,x_0)} \subseteq \overline{Q_{\rho_e R}(t_0,x_0)}$

$\phi|_{Q_{\rho_i R}(t_0,x_0)} = 1$. Posto $W = \phi\omega$

$$\partial_t W_i - \Delta W_i = \partial_i(W_j u_i - W_i u_j) - 2i\,\partial_j(w_i \partial_j \phi)$$
$$+ (\phi_t + \Delta\phi)\, w_i - \partial_j \phi\, (w_j u_i - w_i u_j)$$

Proposition 11.8. *Assume that* $\partial_t W - \Delta W = f$

$$\begin{cases} 1 \le l \le r \le \infty, \quad 1 \le l' \le r' < \infty \\ \frac{d}{l} + \frac{2}{l'} \le \frac{d}{r} + \frac{2}{r'} + 2 \end{cases}$$

or, if $r' = \infty$,

$$\begin{cases} 1 \le l \le r \le \infty, \quad 1 \le l' \le r' = \infty \\ \frac{d}{l} + \frac{2}{l'} < \frac{d}{r} + \frac{2}{r'} + 2. \end{cases}$$

Then there exists a fixed constant $c(d, l, l', r, r')$ *s.t.*

$$\| \int_a^t e^{\triangle(t-t')} f dt' \|_{L^{r'} L^r((a,b)\times\mathbb{R}^d)} < c(d, l, l', r, r') \|f\|_{L^{l'} L^l((a,b)\times\mathbb{R}^d)}.$$

We will use an analogous version involving the gradient of f.

Proposition 11.9. *Assume that* $\partial_t W - \Delta W = \partial f$

$$\begin{cases} 1 \le l \le r \le \infty, \quad 1 \le l' \le r' < \infty \\ \frac{d}{l} + \frac{2}{l'} \le \frac{d}{r} + \frac{2}{r'} + 1 \end{cases}$$

or, if $r' = \infty$,

$$\begin{cases} 1 \le l \le r \le \infty, \quad 1 \le l' \le r' = \infty \\ \frac{d}{l} + \frac{2}{l'} < \frac{d}{r} + \frac{2}{r'} + 1 \end{cases}$$

then there exists a fixed constant $c(d, l, l', r, r')$ *s.t.*

$$\| \int_a^t e^{\triangle(t-t')} \nabla f dt' \|_{L^{r'} L^r((a,b)\times\mathbb{R}^d)} < c(d, l, l', r, r') \|f\|_{L^{l'} L^l((a,b)\times\mathbb{R}^d)}.$$

$$\partial_t W_i - \Delta W_i = \partial_i(W_j u_i - W_i u_j) - 2i\,\partial_j(w_i \partial_j \phi)$$
$$+ (\phi_t + \Delta\phi)\, w_i - \partial_j \phi\, (w_j u_i - w_i u_j)$$

$$|W|_{L^{r'}L^r} \lesssim |Wu|_{L^{a'}L^a} + |w \nabla\phi|_{L^{m'}L^m} + |(\phi_t + \Delta\phi) w|_{L^{e}L^{e'}} + |wu\nabla\phi|_{L^{ee'}}$$

$$\begin{cases} 1 \le a \le r \le \infty \qquad 1 \le a' \le r' \le \infty \\ \frac{3}{a} + \frac{2}{a'} < \frac{3}{r} + \frac{2}{r'} + 1 \end{cases} \qquad \begin{cases} 1 \le m \le r \le \infty \qquad 1 \le m' \le r' \le \infty \\ \frac{3}{m} + \frac{2}{m'} < \frac{3}{r} + \frac{2}{r'} + 1 \end{cases}$$

$$\begin{cases} 1 \le e \le r \le \infty \qquad 1 \le e' \le r' \le \infty \\ \frac{3}{e} + \frac{2}{e'} < \frac{3}{r} + \frac{2}{r'} + 2 \end{cases}$$

$$Q_R(t_0, x_0)$$

$$(t_0 - R^2, t_0] \times \mathbb{R}^3$$

$$(\partial_t - \Delta) W = \partial f \qquad (\partial_t - \Delta) U = f$$

$$W = \nabla U = \nabla \int_0^t e^{(t-t')\Delta} f \, dt' = \int_0^t e^{(t-t')\Delta} \nabla f \, dt'$$

$$\| W \|_{L^{r'} L^{r}} \lesssim \| W u \|_{L^{a'} L^{a}} + \| w \nabla \phi \|_{L^{m'} L^{m}} + \| (\phi_t + \Delta \phi) w \|_{L^{e} L^{e'}} + \| w u \nabla \phi \|_{L^{ee'}}$$

$$\begin{cases} 1 \le a \le r \le \infty & 1 \le a' \le r' \le \infty \\ \frac{3}{a} + \frac{2}{a'} < \frac{3}{r} + \frac{2}{r'} + 1 \end{cases} \qquad \begin{cases} 1 \le m \le r \le \infty & 1 \le m' \le r' \le \infty \\ \frac{3}{m} + \frac{2}{m'} < \frac{3}{r} + \frac{2}{r'} + 1 \end{cases}$$

$$\begin{cases} 1 \le e \le r \le \infty & 1 \le e' \le r' \le \infty \\ \frac{3}{e} + \frac{2}{e'} < \frac{3}{r} + \frac{2}{r'} + 2 \end{cases}$$

$$\frac{1}{e} = \frac{1}{a} = \frac{1}{q} + \frac{1}{m} \qquad \frac{1}{e'} = \frac{1}{a'} = \frac{1}{q'} + \frac{1}{m'}$$

$$\begin{cases} \frac{3}{q} + \frac{3}{m} + \frac{2}{q'} + \frac{2}{m'} < \frac{3}{r} + \frac{2}{r'} + 1 \\ \frac{3}{m} + \frac{2}{m'} < \frac{3}{r} + \frac{2}{r'} + 1 \\ \frac{3}{q} + \frac{3}{m} + \frac{2}{q'} + \frac{2}{m'} < \frac{3}{r} + \frac{2}{r'} + 2 \end{cases}$$

$$\frac{3}{m} - \frac{3}{r} + \frac{2}{m'} - \frac{2}{r'} < 1 - \frac{3}{q} - \frac{2}{q'} \qquad \frac{3}{q} + \frac{2}{q'} < 1$$

$$\chi = \frac{1}{6}\left(1 - \frac{3}{q} - \frac{2}{q'}\right)$$

$$r = \begin{cases} \frac{m}{1 - \chi m} & \text{si } m\chi < 1 \\ \infty & \text{si } m\chi \ge 1 \end{cases} \qquad r' = \begin{cases} \frac{m'}{1 - \chi m'} & m'\chi < 1 \\ \infty & m'\chi \ge 1 \end{cases}$$

$$\frac{3}{m} - \frac{3}{r} \le 3\chi \qquad r = \infty \qquad \frac{1}{m} \le \chi \qquad m\chi \ge 1$$

$$m\chi < 1$$

$$\frac{3}{m} - 3\,\frac{1 - \chi m}{m} = 3\chi \qquad \frac{2}{m'} - \frac{2}{r'} \le 2\chi$$

$$\frac{3}{m} - \frac{3}{r} + \frac{2}{m'} - \frac{2}{r'} \leq 5\chi = \frac{5}{6}\left(1 - \frac{3}{q} - \frac{2}{q'}\right) < 1 - \frac{3}{q} - \frac{2}{q'}$$

$$\Rightarrow \quad W \in L^{r_1'} L^{r_1}(Q_{\rho_i, R}) \Rightarrow w \in L^{r_1'} L^{r_1}(Q_{\rho_i, R})$$

La procedura si ripete finché si arriva alla coppia $(r, r') = (\infty, \infty)$.

Supponiamo di avere iterato k volte $\Rightarrow$

$$w \in L^{r_k' r_k}(Q_{\rho_i^k R})$$

Se $r_k < \infty$ verifichiamo $r_k = \dfrac{m}{1 - k\chi m}$

Lo si verifica per induzione

$$r_1 = \frac{m}{1 - \chi m}$$

$$\vdots$$

$$r_{k-1} = \frac{m}{1 - (k-1)\chi m}$$

$$r_k = \begin{cases} \dfrac{r_{k-1}}{1 - \chi r_{k-1}} & \text{se } r_{k-1}\chi < 1 \\ \infty & \text{se } r_{k-1}\chi \geq 1 \end{cases}$$

$$r_k = \frac{\dfrac{m}{1 - (k-1)\chi m}}{1 - \chi \dfrac{m}{1 - (k-1)\chi m}} = \frac{m}{1 - (k-1)\chi m - \chi m} = \frac{m}{1 - k\chi m}$$

$r_k \chi \geq 1$

$$\frac{m\chi}{1 - k\chi m} \geq 1$$

Alla fine si aggiusta $\rho_i < \rho_e$

$\frac{3}{4} < \rho_i^k$

Prop 11.13 $\exists\ \varepsilon_{q'q} > 0$ t.c. se u è una soluzione debole loc. su $Q_R(t_0, x_0)$ e se soddisfa Serrin con $\frac{2}{q'} + \frac{3}{q} = 1$

e se $\|u\|_{L^{q'}L^{q}(Q_R)} < \varepsilon_{q'q}$

allora u è liscia in $\overline{Q_{R'}(t_0, x_0)}$ $\forall\ 0 < R' < R$.

Si tratta di dimostrare $u \in L^{\beta'}L^{\infty}(Q_{R'})$ con $\beta' > 2$

$\frac{2}{\beta'} + \frac{3}{\infty} < 1$. In realtà si tratta di

dimostrare $w \in L^{\beta'}L^{\infty}(Q_{R'})$

$w \in L^{m'}L^m(Q_R(t_0,x_0))$

$$|W|_{L^{r'}L^r} \lesssim |Wu|_{L^{a'}L^a} + |w\nabla\phi|_{L^{m'}L^m} + |(\phi_t+\Delta\phi)w|_{L^{\ell}L^{\ell'}} + |wu\nabla\phi|_{L^{\ell\ell'}}$$

$$\begin{cases} 1\le a\le r\le\infty \qquad 1\le a'\le r'<\infty \\ \frac{3}{a}+\frac{2}{a'} = \frac{3}{r}+\frac{2}{r'}+1 \quad \checkmark \end{cases}$$

$$\begin{cases} 1\le m\le r\le\infty \qquad 1\le m'\le r'<\infty \\ \frac{3}{m}+\frac{2}{m'} \le \frac{3}{r}+\frac{2}{r'}+1 \end{cases}$$

$$\begin{cases} 1\le \ell\le r\le\infty \qquad 1\le \ell'\le r'<\infty \\ \frac{3}{\ell}+\frac{2}{\ell'} \le \frac{3}{r}+\frac{2}{r'}+2 \end{cases}$$

$$\frac{1}{a} = \frac{1}{q}+\frac{1}{r}, \quad \frac{1}{a'} = \frac{1}{q'}+\frac{1}{r'}$$

$$\frac{1}{\ell} = \frac{1}{q}+\frac{1}{m} \qquad \frac{1}{\ell'} = \frac{1}{q'}+\frac{1}{m'}$$

$$\underbrace{\frac{3}{q}+\frac{2}{q'}}_{=1}+\frac{3}{r}+\frac{2}{r'} = \frac{3}{r}+\frac{2}{r'}+1$$

$$\boxed{\frac{3}{m}+\frac{2}{m'} \le \frac{3}{r}+\frac{2}{r'}+1} \quad \not\checkmark$$

$$\underbrace{\frac{3}{q}+\frac{2}{q'}}_{=1}+\frac{3}{m}+\frac{2}{m'} \le \frac{3}{r}+\frac{2}{r'}+2$$

$$|W|_{L^{r'}L^r} \lesssim |Wu|_{L^{a'}L^a} + |w\nabla\phi|_{L^{m'}L^m} + |(\phi_t+\Delta\phi)w|_{L^{\ell}L^{\ell'}} + |wu\nabla\phi|_{L^{\ell\ell'}}$$

$$|W|_{L^{r'}L^r} \lesssim |W|_{L^{r'}L^r}\,|u|_{L^{q'}L^q} + |w|_{L^{m'}L^m} + |w|_{L^mL^{m'}}\,|u|_{L^{q'}L^q}$$

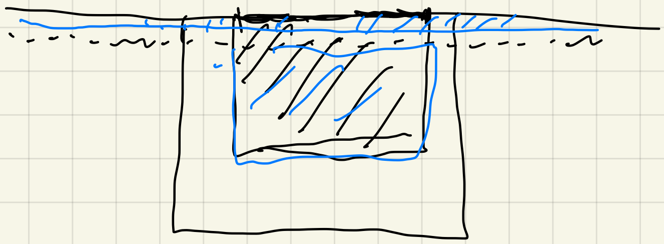
$$|W|_{L^{r'}L^r} \le C_1 |W|_{L^{r'}L^r}\,\varepsilon + C_1|w|_{L^{m'}L^m}$$

$$C_1\varepsilon < \frac{1}{2}$$

$$|W|_{L^rL^r} \le 2C_1 |w|_{L^{m'}L^m}$$

$$\frac{3}{m}-\frac{3}{r}+\frac{2}{m}-\frac{2}{r'} \le 1 \qquad \varkappa = 1$$

Si è dato per scontato che $W \in L^{r'}L^r(Q_R(t_0,x_0))$ $\quad r=\infty \quad r'>2$



$$\partial_t W_i - \Delta W_i = \partial_j (W_j u_i - W_i u_j) - 2\partial_j (w_i \partial_j \phi) + (\partial_t + \Delta)\phi\, w_i - \partial_j \phi\, (w_j u_i - w_i u_j)$$

$$\partial_t W_i^\varepsilon - \Delta W_i^\varepsilon = \partial_j (W_j^\varepsilon u_i^\varepsilon - W_i^\varepsilon u_j^\varepsilon) - 2\partial_j (w_i^\varepsilon \partial_j \phi) + (\partial_t + \Delta)\phi\, w_i^\varepsilon - \partial_j \phi\, (w_j^\varepsilon u_i^\varepsilon - w_i^\varepsilon u_j^\varepsilon)$$

$$(u^\varepsilon, w^\varepsilon) = \rho_\varepsilon * (u, w)$$

$$|W^\varepsilon|_{L^{r'} L^r ((t_0 - R^2, t_0') \times B_R(x_0))} \le C\, |w^\varepsilon|_{L^{m'} L^m ((t_0 - R^2, t_0') \times B_R(x_0))}$$

$$(w^\varepsilon, u^\varepsilon) \xrightarrow{\varepsilon \to 0} (w, u) \quad \text{in } L^{m'} L^m ((t_0 - R^2, t_0') \times B_R(x_0))$$

W^ε è una famiglia limitata $\quad \varepsilon_n \to 0$

$$W^{\varepsilon_n} \rightharpoonup \bar{W} \quad \text{in } L^{r'} L^r ((t_0 - R^2, t_0') \times B_R(x_0))$$

$$|\bar{W}|_{L^{r'} L^r ((t_0 - R^2, t_0) \times B_R(x_0))} \le c\, |w|_{L^{m'} L^m}$$

$t_0' < t_0$

$$\bar{W} = W$$

$$\langle w_i^{\varepsilon}, \psi_i \rangle(t) = \int_{t_0-R^2}^{t} \left(\langle w_i^{\varepsilon}, \partial_t \psi_i \rangle + \langle w_i^{\varepsilon}, \Delta \psi_i \rangle \right) dt'$$

$$- \int_{t_0-R^2}^{t} \langle w_j^{\varepsilon} u_i^{\varepsilon} - w_i^{\varepsilon} u_j^{\varepsilon}, \partial_j \psi_i \rangle \, dt'$$

$$+ 2 \int_{t_0-R^2}^{t} \langle w_i^{\varepsilon} \partial_j \phi, \partial_j \psi_i \rangle \, dt'$$

$$+ \int_{t_0-R^2}^{t} \langle (\phi_t + \Delta \phi) w_i^{\varepsilon}, \partial_j \psi_i \rangle \, dt + \int_{t_0-R}^{t} dt' \langle \partial_j \phi (w_j^{\varepsilon} u_i^{\varepsilon} - w_i^{\varepsilon} u_j^{\varepsilon}), \psi_i \rangle$$