8 dicembre

$$\langle w_i^{\varepsilon_n}, \psi_i\rangle(t) = \int_{t_0-R^2}^{t} \left(\langle w_i^{\varepsilon_n}, \partial_t\psi_i\rangle + \langle w_i^{\varepsilon_n}, \Delta\psi_i\rangle\right) dt'$$

$$-\int_{t_0-R^2}^{t} \langle w_j^{\varepsilon_n}u_i^{\varepsilon_n} - w_i^{\varepsilon_n}u_j^{\varepsilon_n}, \partial_j\psi_i\rangle\, dt'$$

$$+\, 2\int_{t_0-R^2}^{t} \langle w_i^{\varepsilon_n}\partial_j\phi, \partial_j\psi_i\rangle\, dt'$$

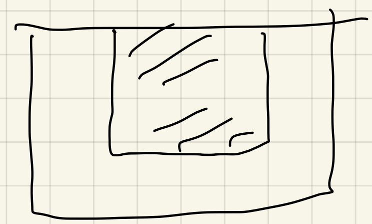
$$+\int_{t_0-R^2}^{t} \langle (\phi_t+\Delta\phi)\, w_i^{\varepsilon_n}, \partial_j\psi_i\rangle\, dt' + \int_{t_0-R}^{t} \langle \partial_j\phi\, (w_j^{\varepsilon_n}u_i^{\varepsilon_n} - w_i^{\varepsilon_n}u_j^{\varepsilon_n}), \psi_i\rangle\, dt'$$

$$\partial_t\bar{w}_i - \Delta\bar{w}_i = \partial_j(\bar{w}_j u_i - \bar{w}_i u_j) - 2\partial_j(w_i\partial_j\phi) + (\partial_t+\Delta)\phi\, w_i - \partial_j\phi\,(w_j u_i - w_i u_j)$$

$(t_0-R, t_0)\times\mathbb{R}^3$

$$\int_{t_0-R^2}^{t} \langle w_i^{\varepsilon_n}, \partial_t\psi_i\rangle\, dt'$$

$\partial_t\psi\in C^0(Q_R(t_0,x_0))$



$w_i^{\varepsilon_n} \rightharpoonup w$ in $L^{r'}L^{r}$

$\partial_t\psi_i \in L^{(r')'}L^{(r)'}$

$\Big\downarrow\ n\to\infty$

$$\int_{t_0-R^2}^{t} \langle \bar{w}, \partial_t\psi\rangle\, dt'$$

$$\int_{t_0-R^2}^{t} \langle w^{\varepsilon_n}, \Delta\psi\rangle\, dt' \longrightarrow \int_{t_0-R^2}^{t} \langle \bar{w}, \Delta\psi\rangle$$

$$\int_{t_0-R^2}^{t} \langle W^{\varepsilon_n} u^{\varepsilon_n}, \nabla \varphi \rangle \, dt' \xrightarrow{n\to\infty} \int_{t_0-R^2}^{t} \langle \bar{W} u, \nabla \varphi \rangle dt'$$

$$\int_{t_0-R^2}^{t} \langle W^{\varepsilon_n} (u^{\varepsilon_n} - u), \nabla \varphi \rangle dt'$$

$$+ \int_{t_0-R^2}^{t} \langle W^{\varepsilon_n} u, \nabla \varphi \rangle \, dt' \longrightarrow \int_{t_0-R^2}^{t} \langle \bar{W} (\varphi, \nabla \varphi) \rangle dt$$

$$\frac{1}{r'} + \frac{1}{q'} \leq 1 \qquad \frac{1}{r} + \frac{1}{q} \leq 1$$

$$\frac{2}{q'} + \frac{3}{q} = 1 \qquad \frac{1}{q'} + \frac{3}{2} \frac{1}{q} = \frac{1}{2}$$

$$\frac{1}{q'} + \frac{1}{q} \leq 1$$

$$\boxed{\frac{1}{q} = \frac{1}{\varphi} + \frac{1}{r} \qquad \frac{1}{q'} = \frac{1}{\varphi} + \frac{1}{r'}}$$

$$\left| \int_{t_0-R^2}^{t} \langle W^{\varepsilon_n}, (u^{\varepsilon_n} - u) \nabla \varphi \rangle dt' \right| \leq$$

$$\leq \| W^{\varepsilon_n} \|_{L^{r'} L^{r} (Q_R(t_0, x_0))} \underbrace{\| u^{\varepsilon_n} - u \|_{L^{q'} L^{q} (Q_R(t_0, x_0))}}_{\downarrow \; 0}$$

$$\| w \|_{L^{m'} L^{m} (Q_R)}$$

$$\lim_{n\to\infty} \langle W_i^{\varepsilon_n}, \psi_i \rangle(t) = \int_{t_0-R^2}^{t} \left(\langle \bar{W}_i, \partial_t \psi_i \rangle + \langle \bar{W}_i, \Delta \psi_i \rangle \right) dt'$$

$$- \int_{t_0-R^2}^{t} \langle \bar{W}_j u_i - \bar{W}_i u_j, \partial_j \psi_i \rangle \, dt'$$

$$+ 2 \int_{t_0-R^2}^{t} \langle \omega_i \partial_j \phi, \partial_j \psi_i \rangle \, dt'$$

$$+ \int_{t_0-R^2}^{t} \langle (\phi_t + \Delta\phi) \omega_i, \partial_j \psi_i \rangle \, dt' + \int_{t_0-R}^{t} \langle \partial_j \phi \, (\omega_j u_i - \omega_i u_j), \psi_i \rangle \, dt'$$

$$\lim_{n\to\infty} \langle W_i^{\varepsilon_n}, \psi_i \rangle(t) = \langle \bar{W}_i, \psi_i \rangle(t)$$

$$W^{\varepsilon_n} \longrightarrow \bar{W} \qquad L^{r'} \quad L^{r}$$

$\psi(t)$

$$\int_{t_0-R^2}^{t_0} \langle W^{\varepsilon_n}, \psi \rangle(t) \, dt \longrightarrow \int_{t_0-R^2}^{t_0} \langle \bar{W}, \psi \rangle(t) \, dt$$

$$\langle W^{\varepsilon_n}, \psi \rangle \longrightarrow \langle \bar{W}, \psi \rangle \qquad \text{in } \boxed{L^1(t_0 - R^2, t_0)}$$

$$\langle W^{\varepsilon_n}, \psi \rangle(t) \longrightarrow \langle \bar{W}, \psi \rangle(t) \qquad \text{q.o. } t.$$

$$\langle \bar{W}_i, \psi_i \rangle(t) = \int_{t_0-R^2}^{t} \left(\langle \bar{W}_i, \partial_t \psi_i \rangle + \langle \bar{W}_i, \Delta \psi_i \rangle \right) dt'$$

$$- \int_{t_0-R^2}^{t} \langle \bar{W}_j u_i - \bar{W}_i u_j, \partial_j \psi_i \rangle \, dt'$$

$$+ 2 \int_{t_0-R^2}^{t} \langle \omega_i \partial_j \phi, \partial_j \psi_i \rangle \, dt'$$

$$+ \int_{t_0-R^2}^{t} \langle (\phi_t + \Delta\phi) \omega_i, \partial_j \psi_i \rangle \, dt' + \int_{t_0-R}^{t} \langle \partial_j \phi \, (\omega_j u_i - \omega_i u_j), \psi_i \rangle \, dt'$$

$\Longrightarrow$ $\bar{W}$ è una soluzione ~~debole~~

$$\partial_t \overline{W}_i - \Delta \overline{W}_i = \partial_j (\overline{W}_j u_i - \overline{W}_i u_j) - 2\partial_j (w_i \partial_j \phi) \qquad (t_0 - R, t_0) \times \mathbb{R}^3$$
$$+ (\partial_t + \Delta)\phi \, w_i - \partial_j \phi \, (w_j u_i - w_i u_j)$$

$$\partial_t W_i - \Delta W_i = \partial_j (W_j u_i - W_i u_j) - 2\partial_j (w_i \partial_j \phi) \qquad (t_0 - R, t_0) \times \mathbb{R}^3$$
$$+ (\partial_t + \Delta)\phi \, w_i - \partial_j \phi \, (w_j u_i - w_i u_j)$$

$$(\partial_t - \Delta)(\overline{W}_i - W_i) = \partial_j \left((\overline{W}_j - W_j) u_i - (\overline{W}_i - W_i) u_j \right)$$

(r', r) $\quad (\varrho, \varrho')$ $\quad \boxed{\frac{3}{\varrho} + \frac{2}{\varrho'} = \frac{3}{r} + \frac{2}{r'} + 1}$

$$|\overline{W} - W|_{L^{r'} L^r} \leq |(\overline{W} - W) u|_{L^{\varrho'} L^{\varrho}} \leq \qquad 1 = \frac{3}{q} + \frac{2}{p'}$$

$$\leq C |\overline{W} - W|_{L^{r'} L^r} \, |u|_{L^{q'} L^q} < C\varepsilon |\overline{W} - W|_{L^{r'} L^r}$$

$$\frac{1}{\varrho} = \frac{1}{r} + \frac{1}{q} \qquad r = r' = 2$$

$$\frac{1}{\varrho'} = \frac{1}{r'} + \frac{1}{q'}$$

$W \in L^2 L^2 (Q_R)$

$\overline{W} \in L^{r'} L^r (Q_R) \Rightarrow$

$r' \geq 2, \; r \geq 2$

$\Downarrow$

$\overline{W} \in L^2 L^2 (Q_R)$

$$\underbrace{(1 - C\epsilon)}_{\geq \frac{1}{2}} |\overline{W} - W|_{L^{r'} L^r} \leq 0$$

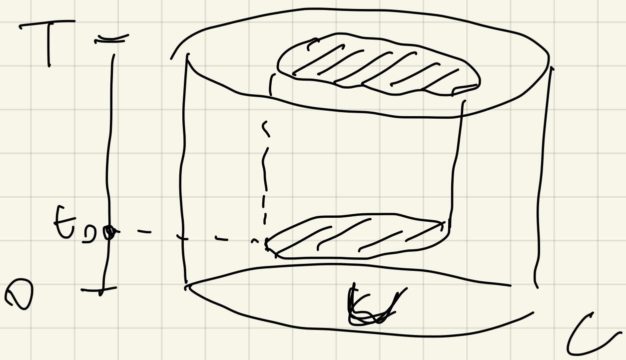
$$\Longrightarrow \quad \overline{W} - W = 0 \text{ in } L^{r'} L^r (Q_R)$$

Teor 11.1 $\quad u$ soluzione debole di NS che soddisfa

in $(0,T)\times U$ locale

$$u \in L^r\big((0,T), L^s(U)\big) \qquad \frac{2}{r}+\frac{3}{s}=1 \qquad (r,s)=(\infty,3)$$

allora



$0<t_0<T$

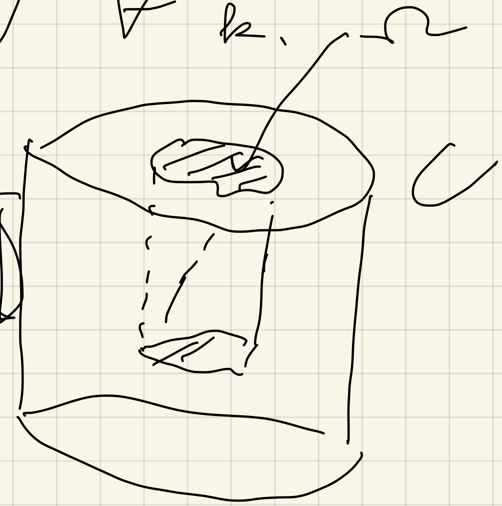
$$u \in C^{0,\gamma}_t\big([t_0,T],\ C^0_x(\bar\Omega)\big)$$

$$0 \le \gamma < \frac{1}{2}$$

$\Omega \subset \overline{\Omega} \subset\subset U \qquad t_0 \in (0,T)$

Le proposizioni garantiscono

$\longrightarrow \boxed{u \in L^\infty([t_0,T], H^k(\Omega))} \quad \forall\, k.$



$\Delta u \in L^\infty([t_0,T], L^\infty(\Omega)) \quad T$

$\underline{u\cdot\nabla u},\ \nabla \underline{p} \in \boxed{L^{\frac{2r}{4r-3}}([t_0,T], L^r(\Omega))}$

$1 < r \le \frac{3}{2}$

$p = R_i R_j (u_i u_j)$

$$\boxed{\partial_t u = -u\cdot\nabla u + \Delta u - \nabla p}$$

$\mathcal{D}'([t_0,T], L^r(\Omega))$

$= \mathcal{L}(\mathcal{D}'(t_0,T), L^r(\Omega))$

$u \in L^\infty([t_0,T], L^r(\Omega))$

$\partial_t u \in L^{\frac{2r}{4r-3}}([t_0,T], L^r(\Omega))$

$u \in W^{1,\frac{2r}{4r-3}}([t_0,T], L^r(\Omega))$

$\left.\frac{2r}{4r-3}\right|_{r=1} = 2 \qquad \left.\frac{2r}{4r-3}\right|_{r=\frac{3}{2}} = \frac{3}{6-3} = 1$

$1 < r \le \frac{3}{2}$

$1 \le \frac{2r}{4r-3} < 2$

$\Rightarrow \quad u \in C^{0,\alpha}([t_0,T], L^r(\Omega))$

$\alpha = 1 - \frac{4r-3}{2r} = \frac{2r-4r+3}{2r} = \frac{3-2r}{2r} > 0$ per $r < \frac{3}{2}$

$$u \in W^{1,\frac{2r}{4r-3}}\left((t_0,T), L^r(\Omega)\right)$$

$$\tilde{u} \in C^{0,\alpha}\left([t_0,T], L^r(\Omega)\right) \qquad \alpha = \frac{3-2r}{2r}$$

$$\tilde{u} = u \qquad 1 < r \le \frac{3}{2}$$

$$u \in C^0\left([0,T], L^2_w(\mathbb{R}^3)\right)$$

$$|u(t)-u(s)|_{L^r(\Omega)} \le C\,|t-s|^\alpha \qquad t,s \in [t_0,T]$$

$$\boxed{|u(t)-u(s)|_{L^\infty(\Omega)} \le C_{\Omega r k}\, |u(t)-u(s)|^{\vartheta}_{L^r(\Omega)}\, |u(t)-u(s)|^{1-\vartheta}_{H^k(\Omega)}}$$

$$\vartheta = \frac{r\left(k-\frac{3}{2}\right)}{kr + \frac{3}{2}(2-r)}$$

$$\cancel{H^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)}$$

$$|f|_{L^\infty} \le C\, |\nabla f|^{\frac{1}{2}}_{L^2}\, |\nabla^2 f|^{\frac{1}{2}}_{L^2}$$

vero per q.o. (t,s) in $[t_0,T]$

$$u \in L^\infty\left([t_0,T], H^k(\Omega)\right)$$

$$|u(t)-u(s)|_{L^\infty(\Omega)} \le C\,|t-s|^{\alpha\vartheta}$$

$$\alpha\vartheta = \frac{3-2r}{2r}\,\frac{r\left(k-\frac{3}{2}\right)}{kr+\frac{3}{2}(2-r)}$$

Qualsiasi $\gamma \in [0, \frac{1}{2})$ è esprimibile come

$$\gamma = \alpha\vartheta$$

$$|u(t)-u(s)|_{C^0(\bar{\Omega})} \le C\,|t-s|^{\gamma}$$

$\exists \quad \tilde{u} = u$ p.o. in t ts.

$$\tilde{u} \in C_t^{0,\gamma}\left([t_0, T], C_x^0(\bar{\Omega})\right)$$

$$u \in C^0\left([t_0, T], L_w^2(\Omega)\right)$$

$$\Rightarrow \quad \tilde{u} = u \, .$$