9 dicembre

Def (Suitable pairs) (u,p) in $(a,b)\times U$

se

1) $u \in L^\infty((a,b), L^2(U))$, $\nabla u \in L^2((a,b)\times U)$,

$p \in L^{\frac{3}{2}}((a,b)\times U, \mathbb{R})$

2) $-\Delta p = \partial_i \partial_j (u_i u_j)$

3) $\forall$ e $\forall\ \phi \in C_c^\infty((a,b)\times U, [0,+\infty))$

$$+2\int_a^{b}\int_{\mathbb{R}^3} |\nabla u|^2 \phi \, dx\, ds \leq$$

$$\leq \int_a^t \int_{\mathbb{R}^3} |u|^2 (\phi_t + \Delta\phi) + \int_a^t \int_{\mathbb{R}^3} (|u|^2 + 2p)(u\cdot\nabla)\phi \, dx\, ds$$

Prop Sia $u_0 \in L^2(\mathbb{R}^3)$ e u una soluzione di Leray-Hopf fornita dalla dimostrazione nel Cap 5, soddisfa la disuguaglianza locale dell'energia in $(0,T)\times\mathbb{R}^3$, con $p = R_i R_j (u_i u_j)$

e $\forall\ \phi \in C_c^\infty((0,T)\times\mathbb{R}^3, [0,+\infty))$

$$2\int_0^T\int_{\mathbb{R}^3} |\nabla u|^2 \phi \, dx\, ds \le$$

$$\le \int_0^T\int_{\mathbb{R}^3} |u|^2(\phi_t + \Delta\phi) + \int_0^T\int_{\mathbb{R}^3} (|u|^2+2p)(u\cdot\nabla)\phi \, dx\, ds$$

Dim se (u,p) è una soluzione classica di NS

$$(\partial_t - \Delta)u + (u\cdot\nabla)u = -\nabla p \qquad \langle \cdot, u\phi\rangle_{L^2_x}$$

$$\int_0^t \Big(\langle \partial_t u, u\phi\rangle - \langle \Delta u, u\phi\rangle + \langle (u\cdot\nabla)u, u\phi\rangle \Big) = -\int_0^t \langle \nabla p, u\phi\rangle$$

$$\int_0^t \langle \partial_t u, u\phi\rangle dt' = \frac{1}{2}\int_0^t \langle \partial_t u^2, \phi\rangle dt' =$$

$$= \frac{1}{2}\langle |u(t)|^2, \phi(t)\rangle - \frac{1}{2}\int_0^t \langle |u(s)|^2, \partial_s\phi\rangle \, ds$$

$$\langle |u(t)|^2, \phi(t)\rangle + 2\int_0^t \Big(\langle \Delta u, u\phi\rangle + \langle (u\cdot\nabla)u, u\phi\rangle \Big) = -2\int_0^t \langle \nabla p, u\phi\rangle$$

$$+ \int_0^t \langle |u|^2, \partial_s \phi\rangle$$

$$2\langle \Delta u, u\phi\rangle = 2\langle \partial_j\partial_j u_k, u_k\phi\rangle =$$

$$= -2\langle \partial_j u_k, \partial_j u_k \phi\rangle - \frac{1}{2}\langle \partial_j |u|^2, \partial_j \phi\rangle$$

$$= -2\langle |\nabla u|^2, \phi\rangle + \cancel{\tfrac{1}{2}}\langle |u|^2, \Delta\phi\rangle$$

$$\langle |u(t)|^2, \phi(t)\rangle + 2\int_0^t \Big(\langle |\nabla u|^2, \phi\rangle + \langle (u\cdot\nabla)u, u\phi\rangle \Big)$$

$$= -2\int_0^t \underbrace{\langle \nabla p, u\phi\rangle} + \int_0^t \langle |u|^2, (\partial_s\phi + \Delta\phi)\rangle$$

$$\langle u_j\partial_j u_k, u_k\phi\rangle = \frac{1}{2}\langle u_j\partial_j |u|^2, \phi\rangle =$$

$$= -\frac{1}{2}\langle |u|^2, (u\cdot\nabla)\phi\rangle$$

$$\int_{\mathbb{R}^3} |u(t)|^2, \phi(t)\, dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2\phi =$$

$$= \int_0^t \int_{\mathbb{R}^3} |u|^2 (\partial_s\phi + \Delta\phi) + \int_0^t \int_{\mathbb{R}^3} \Big(|u|^2 u\cdot\nabla\phi \;\; \overset{2p\, u\cdot\nabla\phi}{-2\nabla p\cdot u\phi} \Big)$$

$$\langle \nabla p, u\phi\rangle = \langle \partial_j p, u_j\phi\rangle = -\langle p, u_j\partial_j\phi\rangle = -\langle p, u\cdot\nabla\phi\rangle$$

$$\begin{cases} (\partial_t - \Delta) u_n + P_n \overset{(u_n\cdot\nabla)u_n}{\operatorname{div}(u_n \otimes u_n)} = -P_n \nabla p_n \\ u_n(0) = P_n u_0 \end{cases}$$

$$p_n = R_i R_j (u_n^i u_n^j) \qquad \langle \ , \varphi u_n \rangle$$

$$\frac{1}{2}\frac{d}{dt}\langle |u_n|^2, \varphi\rangle - \frac{1}{2}\langle |u_n|^2, \partial_t \varphi\rangle + \langle |\nabla u_n|^2, \varphi\rangle - \frac{1}{2}\langle |u_n|^2, \Delta\varphi\rangle$$

$$+ \langle (u_n\cdot\nabla)u_n, \varphi u_n\rangle = -\langle \nabla p_n, \varphi u_n\rangle +$$

$$-\langle (P_n - 1)(u_n\cdot\nabla)u_n, \varphi u_n\rangle + \langle (1-P_n)\nabla p_n, \varphi u_n\rangle$$

$$\boxed{\begin{aligned} 2\int_0^T \langle |\nabla u_n|^2, \varphi\rangle\, dt &= 2\int_0^T \langle |u_n|^2, \partial_t\varphi + \Delta\varphi\rangle \\ &+ \int_0^T \langle |u_n|^2 + 2p_n, (u_n\cdot\nabla)\varphi\rangle \\ &+ \int_0^T \Big(\langle (1-P_n)\nabla p_n, \varphi u_n\rangle + \langle (1-P_n)\, u_n\cdot\nabla u_n, \varphi u_n\rangle\Big) \end{aligned}} \quad \times$$

limite per $n \to +\infty$

$$\int_0^T \langle |u_n|^2, \partial_t\varphi + \Delta\varphi\rangle \longrightarrow \int_0^T \langle |u|^2, \partial_t\varphi + \Delta\varphi\rangle$$

questo perchè $u_n \to u$ in $L^2([0,T]\times K)$

$$\int_0^T \langle |\nabla u_n|^2, \varphi\rangle\, dt$$

$\underline{\nabla u_n} \longrightarrow \nabla u$ in $\underline{L^2([0,\cancel{+\infty}]\times\mathbb{R}^3)}$

$$\int_0^T \langle |\nabla u|^2, \varphi \rangle \, dt \leq \liminf_{n \to \infty} \int_0^T \langle |\nabla u_n|^2, \varphi \rangle \, dt$$

$\varphi \quad dt \quad dx$

$$\lim_n \int_0^T \langle |u_n|^2, (u_n \cdot \nabla)\varphi \rangle \, d\cancel{t} = \int_0^T \langle |u|^2, (u \cdot \nabla)\varphi \rangle \, dt$$

$$\int_0^T \langle |u_n|^2, (u_n \cdot \nabla)\varphi \rangle \, dt - \int_0^T \langle |u|^2, u \cdot \nabla \varphi \rangle \, dt$$

$$\left| \int_0^T \langle u_n - u, u_n (u_n \cdot \nabla)\varphi \rangle \, dt \right| \leq$$

$$\leq |u_n - u|_{L^2((0,T), L^4_x(\Omega))} \, (|u_n|_{L^\infty([0,T], L^2_x(\Omega))}) \, |u_n|_{L^2([0,T], L^4_x(\Omega))}$$

$\xrightarrow{n \to \infty} 0$

$\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$

$\frac{1}{2} + \frac{1}{\infty} + \frac{1}{2} = 1$

$K \supset\supset \Omega$

$$= \left| \, |u_n - u|_{L^4(\Omega)} \right|_{L^2(0,T)} \lesssim \left| \, |u_n - u|^{1-\alpha}_{L^2(\cancel{K})} \, |u_n - u|^{\alpha}_{H^1(\mathbb{R}^3)} \right|_{L^2}$$

$$|u_n - u|^{1-\alpha}_{L^2([0,T] \times K)} \, |u_n - u|^{\alpha}_{L^2((0,T), \cancel{\mathbb{R}} H^1(\mathbb{R}^3))}$$

$\to 0$

$$\int_0^T \langle (1-P_n)\nabla p_n, \varphi u_n \rangle \, dt \xrightarrow{n\to\infty} 0$$

$$\int_0^T \langle (1-P_n)\nabla p_n, \varphi u \rangle \, dt + \int_0^T \langle (1-P_n)\nabla p_n, \varphi (u_n - u) \rangle \, dt$$

$$= \int_0^T \langle \nabla p_n, \underbrace{(1-P_n)(\varphi u)}_{\to 0} \rangle \, dt + \qquad ''$$

$$|\nabla p_n|_{L^2(\mathbb{R}^3)} \; |(1-P_n)\varphi u|_{L^4(\mathbb{R}^3)}$$

$1-P_n \longrightarrow 0$ in senso forte in $L^4(\mathbb{R}^3)$

$1-P_n \quad L^2(\mathbb{R}^3)$

$1-P_n \quad \boxed{L^p(\mathbb{R}^3) \qquad p \neq 2}$

$$|\nabla u_n^2|_{L^2} = |u_n \nabla u_n|_{L^2} \qquad |u_n|_{L^4} \; |\nabla u_n|_{L^2}$$

$P_n^2 \qquad \neq \qquad P_n$

$$P_n = \chi_{[0,n]}(\sqrt{-\Delta}) = \chi_{[-1,1]}\left(\frac{\sqrt{-\Delta}}{n}\right)$$

$$P_n = \chi\left(\frac{\sqrt{-\Delta}}{n}\right) \qquad \chi : C_c^\infty(\mathbb{R}, [0,1])$$

Qui c'è un errore nella costruzione. Prop 12_2 è errata. Prossimo anno modificherò la def dei P_n all'inizio del corso