10 dicembre

Integrazione per parti

Proposizione Dato un intervallo I e $f, g \in C^1(I)$, vale

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx \qquad (1)$$

Date $f, g \in C^1([a,b])$ si ha

$$\int_a^b f'(x)\, g(x)\, dx = f(x)\, g(x)\Big]_a^b - \int_a^b f(x)\, g'(x)\, dx \qquad (2)$$

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx \quad (1)$$

S, note $(fg)' = f'g + fg' \Rightarrow f'g = (fg)' - fg'$

$$\int f'g\,dx = \int \left[(fg)' - fg'\right] dx =$$

$$= \int (fg)'\,dx - \int fg'\,dx$$

$$= fg - \int fg'\,dx$$

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx \qquad (1)$$

$$\int_a^b f'(x)\, g(x)\, dx = \left(\int f'(x)\, g(x)\, dx\right)\Big]_a^b =$$

$$= \left(f(x)\, g(x) - \int f(x)\, g'(x)\, dx\right)\Big]_a^b =$$

$$= f(x)\, g(x)\Big]_a^b - \left(\int f(x)\, g'(x)\, dx\right)\Big]_a^b$$

$$= f(x)\, g(x)\Big]_a^b - \int_a^b f(x)\, g'(x)\, dx$$

$$\int x e^x \, dx = \int x (e^x)' \, dx = x e^x - \int (x)' e^x$$

$$= x e^x - \int e^x = x e^x - e^x + C$$

$$\int x^2 e^x \, dx = \int x^2 (e^x)' \, dx = x^2 e^x - 2\int x e^x \, dx$$

$$= x^2 e^x - 2\left(x e^x - e^x + c \right) =$$

$$= x^2 e^x - 2x e^x + 2 e^x + C$$

$$\int f' g = f g - \int f g'$$

$$\int P(x)\, e^{ax}\, dx$$

$P(x)$ polinomio

$a \neq 0$

$$= \int P(x) \frac{\left(e^{ax}\right)'}{a}\, dx =$$

$$= \frac{1}{a} \int P(x) \left(e^{ax}\right)' dx = \frac{1}{a} P(x)\, e^{ax} - \frac{1}{a} \int P'(x) \frac{\left(e^{ax}\right)'}{a} dx$$

$$= \frac{1}{a} P(x) e^{ax} - \frac{1}{a^2} \left(P'(x)\, e^{ax} - \int P''(x)\, e^{ax}\, dx \right)$$

$$\int e^{ax}\, dx = \frac{e^{ax}}{a} + C$$

$$\int x^2 \sin(x)\,dx = \int x^2 (-\cos x)'\,dx =$$

$$= -x^2 \cos x + 2\int x \cos x\,dx$$

$$= -x^2 \cos x + 2\int x (\sin x)'\,dx$$

$$= -x^2 \cos x + 2x \sin x - 2\int \underbrace{(x)'}_{1} \sin x\,dx$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C$$

$$\int e^x \sin x \, dx = \int (e^x)' \sin x \, dx =$$

$$= e^x \sin x - \int e^x \cos x \, dx$$

$$= e^x \sin x - \int (e^x)' \cos x \, dx$$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$$

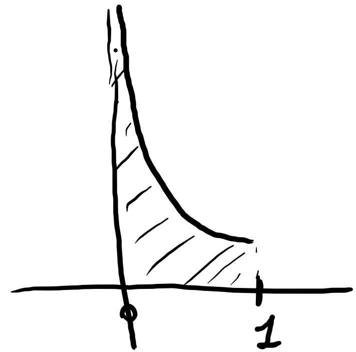
$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + C$$

Integrali impropri

$$\int_0^{+\infty} \frac{1}{1+x^4}\,dx$$

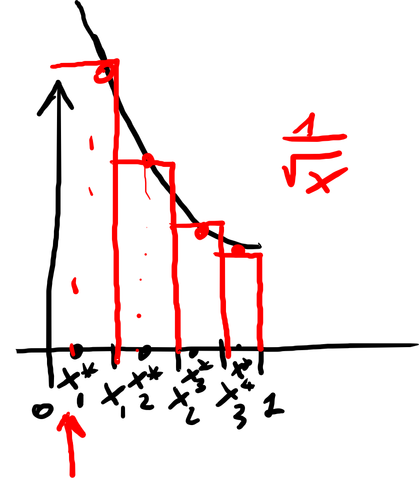
$$\int_0^{1} \frac{1}{\sqrt{x}}\,dx$$



$\frac{1}{\sqrt{x}}$ non è integrabile per Riemann in $[0,1]$.

Infatti se considero somme di Riemann

$\left(f(x_1^*)(x_1-0)\right) + f(x_2^*)(x_2-x_1)$
$+ f(x_3^*)(x_3-x_2) + f(x_4^*)(1-x_3)$



In questo caso, non e' vero che $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

allora $\forall\ \varepsilon>0\ \exists\ \delta_\varepsilon>0$ t.c. per $|\Delta|<\delta_\varepsilon$

$\Delta\quad x_0=0<\dots<x_n=1$

ho $\left|\sum_{j=1}^{n} f(x_j^*)(x_j-x_{j-1}) - \int_0^1 \frac{1}{\sqrt{x}}\,dx\right| < \varepsilon$ $\quad x_j^* \in [x_{j-1}, x_j]$

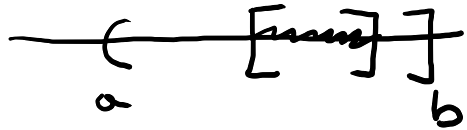
questa e' falsa!

In questo caso non esiste

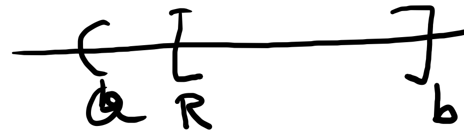
$$\lim_{|\Delta| \to 0} \text{somme di Riemann}$$

Def Sia $f \in L_{loc}((a,b])$ dove $a \in \mathbb{R} \cup \{-\infty\}$

$b \in \mathbb{R}$.



Allora diciamo che f è integrabile (o sommabile) in $(a,b]$ se esiste ed è finito



$$\lim_{R \to a^-} \int_R^b f(x)\,dx ,$$

denotiamo il limite con $\int_a^b f(x)\,dx$

Osservazione Se f è integrabile per Riemann in $[a,b]$, $a, b \in \mathbb{R}$, allora f è integrabile nella definizione appena data.

$[a,b]$

Cioè

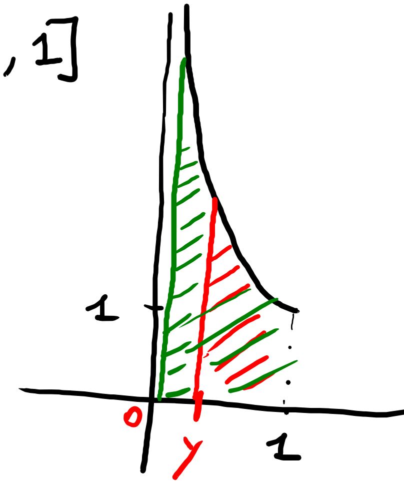
$$\lim_{y\to a^-} \int_y^b f(x)\,dx = \int_a^b f(x)\,dx \Longleftarrow \text{integrale di } R.$$

Questo perchè $\int_b^x f(t)\,dt$ è una funzione continua nella x

$$\Longrightarrow \lim_{y \to a^-} \int_b^y f(t)\,dt = \int_b^a f(t)\,dt$$

$$\Longleftrightarrow \lim_{y \to a^-} \int_y^b f(t)\,dt = \int_a^b f(t)\,dt.$$

Esempio $\dfrac{1}{x^p}$ $p > 0$ in $(0, 1]$



Sono integrabili esattamente per

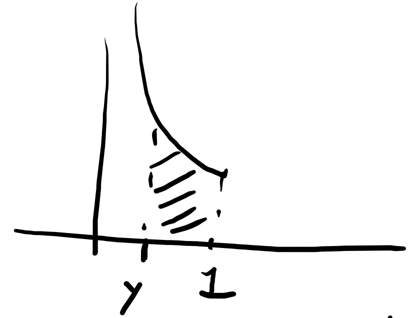
$0 < p < 1$

Invece per $p \geq 1$ non sono integrabili

$p \neq 1$ $\quad 0 < y < 1$

$$\int_y^1 x^{-p}\,dx = \frac{x^{1-p}}{1-p}\Bigg]_y^1 = \frac{1}{1-p} - \frac{y^{1-p}}{1-p}$$

$$\int_y^1 x^{-p}\,dx = \frac{1}{1-p} - \frac{y^{1-p}}{1-p}$$



$p \neq 1$

$p > 0$

$$\lim_{y\to 0^+} \int_y^1 x^{-p}\,dx =$$

$$= \frac{1}{1-p} - \lim_{y\to 0^+} \frac{y^{1-p}}{1-p}$$

$$\lim_{y\to 0^+} \frac{y^{1-p}}{1-p} = \begin{cases} 0 & \text{se } p<1 \\ -\infty & \text{se } p>1 \end{cases}$$

<u>Esercizio</u> Sia $f \in L[a,b]$ e sia $[\alpha,\beta] \subset [a,b]$

Allora $f \in L[\alpha,\beta]$.

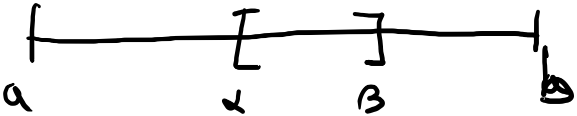
<u>Dim</u> $f \in L[a,b] \Longleftrightarrow$

$\forall \varepsilon > 0 \quad \exists \Delta_\varepsilon$ (decomp di $[a,b]$) t.c.

$0 \leq S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon$.

Possiamo supporre, eventualmente prendendo un raffinamento di Δ_ε, che i punti α e β

Prendiamo Δ_ε $\quad x_0 = a < x_1 < \dots < \underset{\underset{\alpha}{\parallel}}{x_{k_0}} < \dots < \underset{\underset{\beta}{\parallel}}{x_{h_0}} < \dots < x_n = b$



Notiamo che resta definita una decomposizione di $[\alpha, \beta]$

$\tilde{\Delta}_\varepsilon \qquad x_{k_0} = \alpha < \dots < x_{h_0} = \beta$

$$0 \le S(\Delta_\varepsilon) - s(\Delta_\varepsilon) = \sum_{j=1}^{n} \overbrace{(x_j - x_{j-1})}^{\ge 0} \left(\overbrace{\sup f[x_{j-1}, x_j] - \inf f[x_{j-1}, x_j]}^{\ge 0} \right) < \varepsilon$$

$$\Rightarrow 0 \le \underbrace{\sum_{j=k_0+1}^{h_0} (x_j - x_{j-1}) \left(\sup f[x_{j-1}, x_j] - \inf f[x_{j-1}, x_j] \right)}_{S(\tilde{\Delta}_\varepsilon) - s(\tilde{\Delta}_\varepsilon)} < \varepsilon$$

Quindi abbiamo dimostrato che $\forall \varepsilon > 0$ $\exists\ \tilde{\Delta}_\varepsilon$ decomp di $[\alpha, \beta]$ t.c.

$$0 \leq S(\tilde{\Delta}_\varepsilon) - s(\tilde{\Delta}_\varepsilon) < \varepsilon$$

$$\Longrightarrow \quad f \in L[\alpha, \beta].$$