Markov chains

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Markov chains

- Walker: mathematical entity whose attributes completely define the state of the system
- The walker moves in a given space by a combination of deterministic and random displacements
- Consider a discrete system of W available states, named S_1 through S_W
- At every discrete point i, $x^{(i)} = S_j$ if the system is in the state S_j at step i

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- The probability of the system changing from state S_j to S_k in one step is given by

$$P_{kj} \equiv P(x^{i+1} = S_k \leftarrow x^i = S_j)$$

- Normalization:

$$\sum_{k=1}^{W} P_{kj} = 1$$

- $p_k^{(i)} \rightarrow$ probability that the system is in state S_k at time i
- Vector representation of p

$$\mathbf{p}^{(i)} = \begin{bmatrix} p_1^{(i)} \\ . \\ . \\ . \\ p_W^{(i)} \end{bmatrix}, \sum_{k}^{W} p_k^{(i)} = 1$$

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Evolution of p:

$$
\begin{aligned}
p_k^{(i+1)} &= \sum_j P_{kj} p_j^{(i)} \\
\mathbf{p}^{(i+1)} &= \mathbf{P}\mathbf{p}^{(i)} \\
\mathbf{p}^{(1)} &= \mathbf{P}\mathbf{p}^{(0)} \\
\mathbf{p}^{(2)} &= \mathbf{P}\mathbf{p}^{(1)} = \mathbf{P}\mathbf{P}\mathbf{p}^{(0)} \\
\mathbf{p}^{(m)} &= \mathbf{P}^m\mathbf{p}^{(0)}
\end{aligned}
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- After a sufficiently long time M, $|\mathbf{p}^{(M+1)} - \mathbf{p}^{(M)}| \to 0$
- Equilibrium probability distribution $\mathbf{p}^*$

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Numerical example: three state Markov process with

$$\mathbf{P} = \begin{bmatrix} 1/4 & 1/8 & 2/3 \\ 3/4 & 5/8 & 0 \\ 0 & 1/4 & 1/3 \end{bmatrix}$$

and

$$\mathbf{p}^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The first three steps are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1/4 \\ 3/4 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 5/32 \\ 21/32 \\ 6/32 \end{bmatrix}$$

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Iteration	p_1	p_2	p_3
0	1.00000	0.00000	0.00000
1	0.25000	0.75000	0.00000
2	0.15625	0.65625	0.18750
3	0.24609	0.52734	0.22656
4	0.27848	0.51416	0.20736
5	0.27213	0.53021	0.19766
6	0.26608	0.53548	0.19844
7	0.26575	0.53424	0.20002
8	0.26656	0.53321	0.20023
9	0.26678	0.53318	0.20005
10	0.26671	0.53332	0.19998
11	0.26666	0.53335	0.19999
12	0.26666	0.53334	0.20000
13	0.26667	0.53333	0.20000
$\mathbf{p}^*$	0.26667	0.53333	0.20000

- Steady solution of $\mathbf{P}$

$$
\begin{aligned}
p_1^* &= 1/4p_1^* + 1/8p_2^* + 2/3p_3^* \\
p_2^* &= 3/4p_1^* + 5/8p_3^* \\
p_3^* &= 1/4p_2^* + 1/3p_3^*
\end{aligned}
$$

- with the constraint $p_1^* + p_2^* + p_3^* = 1$
- The final (steady) state is given by

$$
\mathbf{p}^* = \begin{bmatrix} 4/15 \\ 8/15 \\ 3/15 \end{bmatrix}
$$