11 dicembre

Lemma (cambio var. int ind.) Siano I e J due intervalli, $u: I \to J$, $f(u): J \to \mathbb{R}$ con $u \in C^1(I)$ e $f \in C^0(J)$. Denotiamo con $\int f(u)\,du$ le primitive di $f(u)$. Allora

$$(1) \quad \int f(u(x))\, u'(x)\, dx = \left(\int f(u)\,du\right)(\underline{u(x)})$$

<u>Osservazione</u> Di solito (1) viene scritto nella forma

$$\int f(u(x))\, u'(x)\, dx = \underline{\int f(u)\,du} \qquad (2)$$

<u>Dim</u> $\int f(u(x))\, u'(x)\, dx = \left(\int f(u)\,du\right)(u(x))$

$\frac{d}{dx} \int f(u(x))\, u'(x)\, dx = \underline{f(u(x))\, u'(x)}$

per def di primitiva

Notiamo $\frac{d}{du} \int f(u)\,du = f(u)$ di nuovo per la def di primitiva.

$$\frac{d}{dx}\left(\int f(u)\,du\right)(u(x)) =$$

$$= \frac{d}{du}\left(\int f(u)\,du\right)(u(x)) \qquad \frac{d}{dx}u(x)$$

$$= \qquad f(u(x)) \quad u'(x)$$

La (1) è dimostrata. $\square$

Esempi

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x}\, dx$$

. Ricordiamo

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

$u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int \frac{-\sin x \, dx}{\cos x} = -\int \frac{du}{u}$$

$$= -\log|u| + C = -\log|\cos x| + C$$

$$\int \sqrt{ax+b}\,dx \qquad u = ax+b \quad du = a\,dx$$

$$= \frac{1}{a}\int \sqrt{u}\;du =$$

$$= \frac{1}{a}\int u^{\frac{1}{2}}du = \frac{1}{a}\;\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c =$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + c$$

$$\int \cos^{2n+1}(x)\,\sin^{m}(x)\,dx =$$

$$= \int \cos^{2n}(x)\,\sin^{m}(x)\,\cos x\,dx$$

Poi usiamo $\cos^2(x) = 1 - \sin^2(x)$

$$= \int (1-\sin^2(x))^n \sin^m(x)\,\cos(x)\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int (1-u^2)^n\,u^m\,du$$

Ad es, $n = 1 \quad m = 3$

$$\int \cos^3(x) \sin^3(x)\, dx = \qquad \boxed{u = \sin x}$$

$$= \int (1-u^2)\, u^3\, du$$

$$= \int (u^3 - u^5)\, du = \frac{u^4}{4} - \frac{u^6}{6} + c$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c$$

$$\int \sin^{2n}(x) \cos^{2m}(x)\, dx =$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^n \left(\frac{1+\cos(2x)}{2}\right)^m dx$$

$$= \frac{1}{2^{n+m}} \int (1-\cos(2x))^n (1+\cos(2x))^m\, dx$$

$$y = 2x$$

$$dy = 2\, dx$$

$$= \frac{1}{2^{n+m+1}} \int (1-\cos(y))^n (1+\cos(y))^m\, dy$$

$$\int \cos^4(x) \sin^2(x)\, dx =$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 \frac{1-\cos(2x)}{2}\, dx$$

$$= \frac{1}{8} \int (1+\cos(2x)) (1-\cos^2(2x))\, dx$$

$$= \frac{1}{8} \int \left(1 - \cos^2(2x) + \cos(2x) - \cos^3(2x)\right) dx$$

$$\int \cos(2x)\, dx = \frac{\sin(2x)}{2} + c$$

$$\int \cos^3(2x)\, dx = \int (1-\sin^2(2x)) \cos(2x)\, dx$$

$u = \sin(2x)$

$du = \cos(2x) \cdot 2 \cdot dx$

$$= \frac{1}{2} \int (1-u^2)\, du = \frac{u}{2} - \frac{u^3}{6} + c \qquad u = \sin(2x)$$

$$\int \cos^2(2x)\, dx = \int \frac{1+\cos(4x)}{2}\, dx =$$

$$= \frac{x}{2} + \frac{1}{2} \int \cos(4x)\, dx = \frac{x}{2} + \frac{1}{2} \frac{\sin(4x)}{4} + C$$

$f \in L_{loc}((a,b])$ si dice integrabile in $(a,b]$, in senso improprio, se $\lim_{y \to a^+} \int_y^b f(x)\,dx$ esiste ed è finito.

$f \in L_{loc}([a,b))$ f si dice integrabile in $[a,b)$, in senso improprio se

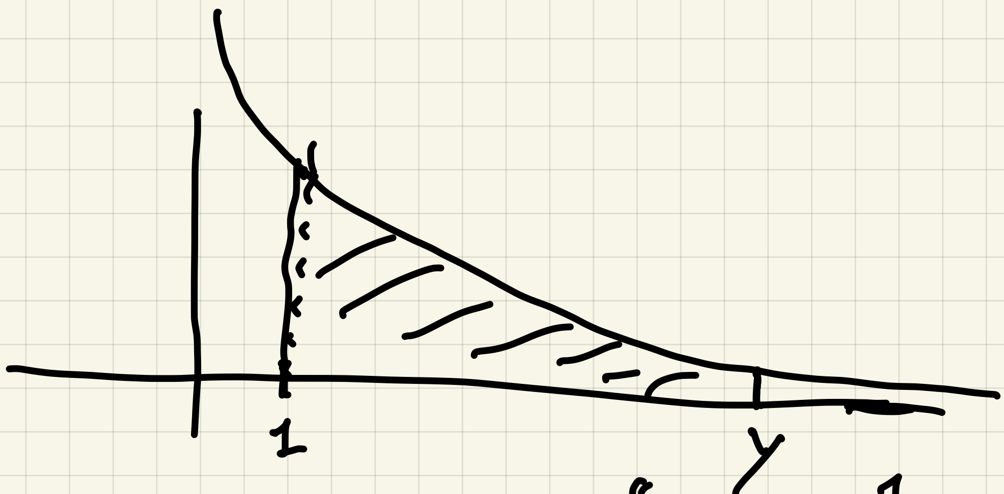
$$\lim_{y \to b^-} \int_a^y f(x)\,dx$$ esiste ed è finito

<u>Esempio</u> Le funzioni x^{-p} sono integrabili in $[1,+\infty)$ se e solo se $p > 1$.

$p \neq 1$

$$\lim_{y\to+\infty}\int_1^y x^{-p}\,dx$$

$$\int_1^y x^{-p}\,dx = \left.\frac{x^{1-p}}{1-p}\right]_1^y = \frac{y^{1-p}}{1-p} - \frac{1}{1-p}$$



$$\lim_{y\to+\infty}\left(\frac{y^{1-p}}{1-p}-\frac{1}{1-p}\right)=\begin{cases} -\dfrac{1}{1-p}=\dfrac{1}{p-1} & \text{se } p>1\\ +\infty & p<1\end{cases}$$

$p = 1$

$$\lim_{y\to+\infty}\int_1^y \frac{1}{x}\,dx = \lim_{y\to+\infty} \left.\lg x\right]_1^y =$$

$$= \lim_{y\to+\infty} \lg y = +\infty$$

$$\int_2^{+\infty} \frac{1}{x \lg x}\, dx$$

$$\int_2^{R} \frac{1}{x \lg x}\, dx =$$

$y = \lg x$

$dy = \frac{1}{x}\, dx$

$$= \int_{\lg 2}^{\lg R} \frac{1}{y}\, dy \xrightarrow{R \to +\infty} +\infty$$

$$\frac{1}{x \lg x} \notin L[2, +\infty)$$

Più in generale $\frac{1}{x \lg^p x} \in L([2, +\infty))$

se e solo se $p > 1$

$$\int_2^{R} \frac{1}{x \lg^p x}\, dx$$

$y = \lg x$

$dy = \frac{1}{x}\, dx$

$$= \int_{\lg 2}^{\lg R} \frac{1}{y^p}\, dx = \frac{y^{-p+1}}{-p+1} \Big]_{\lg 2}^{\lg R}$$

$$\int_2^R \frac{1}{x \lg^p x}\, dx \qquad\qquad y = \lg x \qquad p>1$$

$$dy = \frac{1}{x}\, dx$$

$$= \int_{\lg 2}^{\lg R} \frac{1}{y^p}\, dx = \left.\frac{y^{-p+1}}{-p+1}\right]_{\lg 2}^{\lg R}$$

$$\lim_{R\to+\infty} \left(\frac{\lg^{\overset{<0}{1-p}} R}{1-p} - \frac{\lg^{1-p} 2}{1-p} \right) =$$

$$= - \frac{\lg^{1-p} 2}{1-p} = \frac{\lg^{1-p} 2}{p-1} \qquad p>1$$

$$\frac{1}{x \lg x \ \lg^p(\lg x)} \in L([10,\infty))$$

$$\Longleftrightarrow \quad p>1$$

$L([K,+\infty))$ nu

$K>>1 \Longleftrightarrow p>1$

$$\frac{1}{x \lg x \lg(\lg x) \dots \dots \underbrace{\lg(\lg(\lg(\dots \lg(x))}_{n}\ \underbrace{\lg^p(\lg(\lg \dots\dots \lg(x))}}$$

n volte

Teor 1) Se $f, g \in L([a,b))$ e se $\lambda, \mu \in \mathbb{R}$ allora

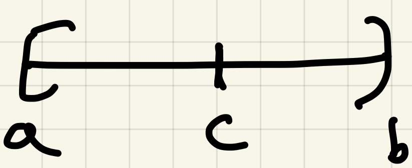
$$\int_a^b (\lambda f + \mu g)\, dx = \lambda \int_a^b f \, dx + \mu \int_a^b g \, dx$$

2) Se $f, g \in L([a,b))$, se $f(x) \leq g(x)$ $\forall x$ in $[a,b)$ $\Rightarrow$ $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$

e se $f, g \in C^0([a,b))$ e non si ha $f(x) = g(x)$ $\forall x$, $\Rightarrow$ $\int_a^b f(x)\, dx < \int_a^b g(x)\, dx$.

3) Se $c \in (a,b)$, $f \in L_{loc}[a,b)$, allora

$f \in L[a,b) \Longleftrightarrow f \in L[c,b)$



e inoltre vale

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$