$$a^n - b^n = (a-b) \sum_{j=1}^{n} a^{n-j} b^{j-1}$$

Per induzione

1) $n = 1$ a posto

2) assumiamo $n-1$

$$a^{n-1} - b^{n-1} = (a-b) \sum_{j=1}^{n-1} a^{n-1-j} b^{j-1} \qquad (n-1)$$

$$(a-b) \sum_{j=1}^{n} a^{n-j} b^{j-1} =$$

$$= (a-b) \left(\sum_{j=1}^{n-1} a^{n-j} b^{j-1} + b^{n-1} \right)$$

$$= (a-b) \sum_{j=1}^{n-1} a^{n-j} b^{j-1} + (a-b) b^{n-1}$$

$$= (a-b) \sum_{j=1}^{n-1} a \, a^{n-1-j} b^{j-1} + (a-b) b^{n-1}$$

$$= a \underbrace{(a-b) \sum_{j=1}^{n-1} a^{n-1-j} b^{j-1}} + (a-b) b^{n-1}$$

$$= a \left(a^{n-1} - b^{n-1} \right) + (a-b) b^{n-1}$$

$$= a^n - \cancel{b^{n-1} a} + \cancel{a b^{n-1}} - b^n = a^n - b^n$$

$$\lg(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\lg(1+x+x^2) = x+x^2 - \frac{(x+x^2)^2}{2} + o\left((x+x^2)^2\right)$$

$$(x+x^2)^2 = x^2(1+x)^2 = x^2(1+o(1))$$

$$o\left(x^2(1+o(1))\right) = o(x^2)$$

$$\lg(1+x+x^2) = x+x^2 - \frac{(x+x^2)^2}{2} + o(x^2)$$

$$= x+x^2 - \frac{x^2+2x^3+x^4}{2} + o(x^2)$$

$$= x+x^2 - \frac{x^2}{2} + o(x^2) = x + \frac{x^2}{2} + o(x^2)$$

$$\boxed{o\left((x+x^2)^2\right) = o(x+x^2)}$$

$$\boxed{o(\cancel{y}^2) = o(y)}$$

$$\lim_{y\to 0} \frac{o(y^2)}{y^2} = 0 \Rightarrow \lim_{y\to 0} \frac{o(y^2)}{y} = \lim_{y\to 0}\left(\frac{o(y^2)}{y^2}\right)y$$

$$= 0 \cdot 0 = 0$$

$$\lim_{x\to 0} \frac{o\left(x^2(1+o(1))\right)}{x^2} =$$

$$= \lim_{x\to 0}\left(\frac{o\left(x^2(1+o(1))\right)}{x^2(1+o(1))}\right)(1+o(1)) = 0 \cdot 1 = 0$$

$$o\left((x+x^2)^2\right) = o\left(x^2+2x^3+x^4\right)$$

$$\lim_{x\to 0} \frac{o(x^2+2x^3+x^4)}{x^2} =$$

$$= \lim_{x\to 0} \underbrace{\frac{o(x^2+2x^3+x^4)}{x^2+2x^3+x^4}}_{\downarrow \atop 0} \; \underbrace{\frac{x^2+2x^3+x^4}{x^2}}_{\downarrow \atop 1} = 0$$

$$2\ \frac{x^a - \frac{1}{2}x^{2a} - x^2 + o(x^{2a})}{x^2}$$ per $a<$

$$x^a - \frac{1}{2}x^{2a} - x^2 + o(x^{2a}) = \begin{cases} x^a + o(x^a) & \text{se } 0<a<2 \\ -x^2 + o(x^2) & a>2 \\ -\frac{1}{2}x^4 + o(x^4) & a=2 \end{cases}$$

$$\lim_{x\to+\infty}\left(x^3 - x^2\right) = \lim_{x\to+\infty} x^3 = +\infty$$

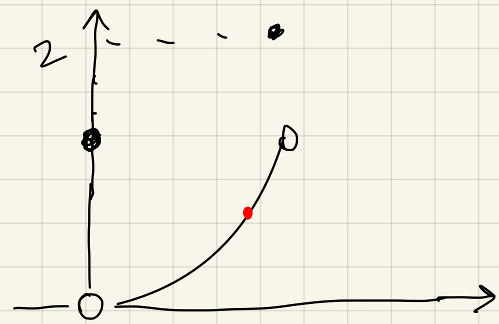
$a<2$ $\quad \lim_{x\to 0^+} 2\frac{x^a}{x^2} = +\infty$

$a=2$ $\quad \lim_{x\to 0^+} 2\frac{\left(-\frac{1}{2}\right)x^4}{x^2} = 0$

$a>2$ $\quad \lim_{x\to 0^+} 2\ \frac{-x^2}{x^2} = -2$

f convessa tra $[0,1]$

$$f = \begin{cases} 1 & x = 0 \\ x^2 & 0 < x < 1 \\ 2 & x = 1 \end{cases}$$



È convessa in $(0,1)$, $f''(x) = 2 > 0$

$$\boxed{f((1-t)x_0 + t x_1) \leq (1-t) f(x_0) + t f(x_1)} \quad \forall x_0, x_1 \in [0,1] \text{ e } \forall t \in (0,1)$$

Se $x_0, x_1 \in (0,1)$ sappiamo che è vero

Sia ora $x_0 = 0$ e $x_1 \in (0,1]$ $\quad t \in (0,1)$

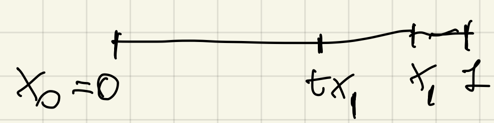
$$f((1-t)\underbrace{x_0}_{=0} + t x_1) =$$

$$= ((1-t)x_0 + t x_1)^2 \leq$$

$$\leq (1-t)\, 0^2 + t x_1^2$$

$$\leq (1-t) f(0) + t x_1^2$$

$$\leq (1-t) f(0) + t f(x_1)$$



$f(0) \geq 0$

$f(x_1) \geq x_1^2$

$0, \; x_1 \neq 0$

$\mathbb{R} = \mathbb{Q}'$

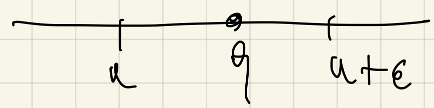
Sia $a \in \mathbb{R}$. Allora $a \in \mathbb{Q}'$ se

$\forall \varepsilon > 0 \ \exists\, q \in \mathbb{Q}$ t.c. $0 < |a - q| < \varepsilon$.

Essendo $\mathbb{Q}$ denso in $\mathbb{R}$, per ogni $\underline{b} > a$ $\exists$ ~~esiste~~ $q \in \mathbb{Q}$ t.c. $a < q < \underline{b}$.

$$0 < q - a < b - a = \varepsilon \qquad b = a + \varepsilon$$

$$0 < q - a < \varepsilon.$$



Se $f \in C^0(\mathbb{R})$ con $\lim_{x\to\infty} f(x)=+\infty$

allora f non ha punti di massimo ma ha punti di minimo assoluto.

1) Non punti di massimo assoluto perché se per assurdo

Se per assurdo x_M fosse un punto di massimo assoluto ~~...~~, allora avrei $\boxed{f(x_M)\ge f(x)\ \forall x\in\mathbb{R}}$

Siccome $\lim_{x\to\infty} f(x)=+\infty$

$$\boxed{\forall K\in\mathbb{R}\ \exists M_K \text{ t.c. } |x|\ge M_K \Rightarrow f(x)> K}$$

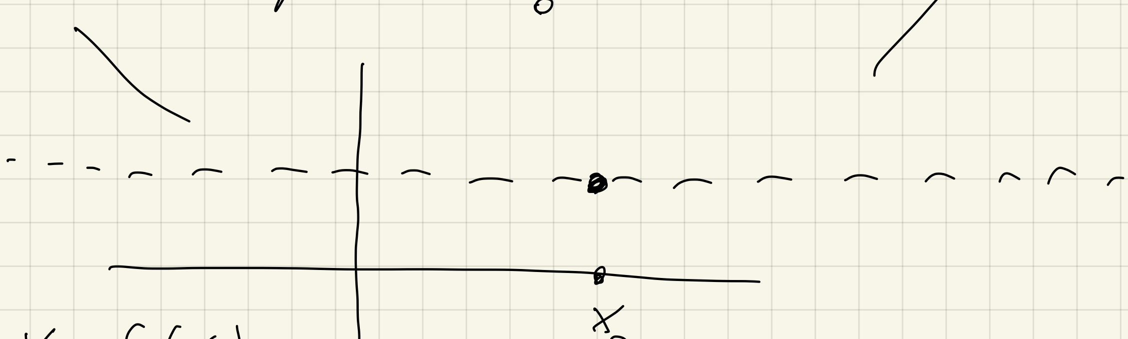
scelto $K=f(x_M)$

allora per $|x|\ge M_K \Rightarrow f(x)>f(x_M)\ge f(x)$ assurdo

2) $\lim_{x\to\infty} f(x)=+\infty$ significa che

$$\boxed{\forall K\ \exists M_K \text{ t.c. } |x|> M_K \Rightarrow f(x)> K}.$$

Fissiamo un qualsiasi $x_0\in\mathbb{R}$



Sia $K=f(x_0)$

Allora per $|x|>M_K \Rightarrow f(x)>f(x_0)$

$\Rightarrow |x_0|\le M_K \iff x_0\in[-M_K,M_K]$

~~Ax~~ $f \in C^0([-M_K, M_K])$ $\quad K = f(x_0)$

Per Weierstrass in $[-M_K, M_K]$ la f ha un punto di minimo assoluto, x_m.

Osserviamo che $x_0 \in [-M_K, M_K]$

x_m p.t. di minimo in $[-M_K, M_K]$ significa

$$\boxed{f(x_m) \leq f(x) \quad \forall x \in [-M_K, M_K]}. \quad *$$

In particolare, $f(x_m) \leq f(x_0)$.

Per $\boxed{\begin{array}{l} x \notin [-M_K, M_K] \iff |x| > M_K \\ f(x) > f(x_0) (= K) \geq f(x_m) \end{array}}$

$\Rightarrow$

Conclusione

$$\forall x \in \mathbb{R} \text{ si ha } f(x_m) \leq f(x)$$

$\Rightarrow$ x_m e' un pt di min. assoluto in $\mathbb{R}$.