14 dicembre

Prop 12.2 Sia $u \in L^\infty((a,b), L^2(U))$ e $\nabla u \in L^2((a,b)\times U)$ t.c.

$$2\int_a^b \int_{\mathbb{R}^3} |\nabla u|^2 \varphi \, dx\, ds \le \int_a^b \int_{\mathbb{R}^3} |u|^2 (\varphi_s + \Delta\varphi)\, dx\, ds$$

$$+ \int_a^b \int_{\mathbb{R}^3} (|u|^2 + 2p)(u\cdot\nabla)\varphi \, dx\, ds \qquad \forall \varphi \in C_c^\infty((a,b)\times U, [0,+\infty))$$

Allora $\forall\ t \in (a,b)$ vale

$$\int_{\mathbb{R}^3} |u(t)|^2 \phi(t)\, dx + $$

$$+ 2\int_a^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \, dx\, ds \le \int_a^{t} \int_{\mathbb{R}^3} |u|^2 (\phi_s + \Delta\phi)\, dx\, ds$$

$$+ \int_a^{t} \int_{\mathbb{R}^3} (|u|^2 + 2p)(u\cdot\nabla)\phi \, dx\, ds$$

$$\forall \phi \in C_c^\infty((a,b)\times U, [0,+\infty))$$

Dim

$$\varphi_\varepsilon(s,x) = \phi(s,x)\, \chi\left(\frac{t-s}{\varepsilon}\right)$$

$\chi = 0$ in $\mathbb{R}_-$, 1 in $[1,+\infty)$

$\chi'\left(\frac{t-s}{\varepsilon}\right) = 0$ in $s \le t-\varepsilon$ e $s \ge t$

$$\int_{t-\varepsilon}^{t} \varepsilon^{-1}\chi'\left(\frac{t-s}{\varepsilon}\right)ds = -\chi\left(\frac{t-s}{\varepsilon}\right)\Big]_{t-\varepsilon}^{t} =$$

$$= -\chi(0)+\chi(1) = 1$$

$$\partial_s \varphi_\varepsilon(s,x) = \partial_s\left(\chi\left(\frac{t-s}{\varepsilon}\right)\phi(s,x)\right) =$$

$$= \partial_s\phi(s,x)\,\chi\left(\frac{t-s}{\varepsilon}\right) - \phi(s,x)\,\varepsilon^{-1}\chi'\left(\frac{t-s}{\varepsilon}\right)$$

$$2\int_a^b\int_{\mathbb{R}^3}|\nabla u|^2\varphi_\varepsilon\,dx\,ds \le \int_a^b\int_{\mathbb{R}^3}|u|^2\left(\varphi_{\varepsilon s}+\Delta\varphi_\varepsilon\right)dx\,ds$$

$$+\int_a^b\int_{\mathbb{R}^3}\left(|u|^2+2p\right)(u\cdot\nabla)\varphi_\varepsilon\,dx\,ds$$

$$2\int_a^t\int_{\mathbb{R}^3}|\nabla u|^2\,\chi\left(\frac{t-s}{\varepsilon}\right)\phi(s,x)\,ds\,dx$$

$$+\int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3}|u|^2\,\phi(s,x)\,\varepsilon^{-1}\chi'\left(\frac{t-s}{\varepsilon}\right) \le$$

$$\le \int_a^t\int_{\mathbb{R}^3}|u|^2\left(\partial_s\phi(s,x)+\Delta\phi(s,x)\right)\chi\left(\frac{t-s}{\varepsilon}\right)$$

$$+\int_a^t\int_{\mathbb{R}^3}\left(|u|^2+2p\right)u\cdot\nabla\phi(s,x)\,\chi\left(\frac{t-s}{\varepsilon}\right).$$

$$2\int_a^t \int_{\mathbb{R}^3} |\nabla u|^2 \quad \phi(s,x)\, ds\, dx \quad +$$

$$\lim_{\varepsilon\to 0^+} \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} |u|^2 \quad \phi(s,x) \quad \varepsilon^{-1}\chi'\left(\frac{t-s}{\varepsilon}\right) \leq$$

$$\leq \int_a^t\int_{\mathbb{R}^3} |u|^2 \left(\partial_s\phi(s,x)+\Delta\,\phi(s,x)\right)$$

$$+\int_a^t\int_{\mathbb{R}^3}\left(|u|^2+2p\right)\, u\cdot\nabla\,\phi(s,x)$$

R

$$\lim_{\varepsilon\to 0^+} \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} |u|^2 \quad \phi(s,x) \quad \varepsilon^{-1}\chi'\left(\frac{t-s}{\varepsilon}\right) =$$

$$= \lim_{\varepsilon\to 0^+} \left(\int_{t-\varepsilon}^t\right) \varepsilon^{-1}\chi'\left(\frac{t-s}{\varepsilon}\right) \int_{\mathbb{R}^3}|u(s,x)|^2\phi(s,x)\,dx$$

$$s \longrightarrow \underbrace{\int_{\mathbb{R}^3} |u(s,x)|^2 \phi(s,x)\, dx}_{\mathcal{L}(s)} \in L^p(\mathbb{R}) \quad \forall p$$

$$p<+\infty$$

$$\int_{t-\varepsilon}^t \varepsilon^{-1}\chi'\left(\frac{t-s}{\varepsilon}\right)\mathcal{L}(s)\, ds \xrightarrow{\varepsilon\to 0^+} \mathcal{L}(s) \quad \text{in } L^p(\mathbb{R})$$

$$\varepsilon_n \to 0^+$$

$$\int_{t-\varepsilon_n}^t \varepsilon_n^{-1}\chi'\left(\frac{t-s}{\varepsilon_n}\right)\mathcal{L}(s)\, ds \xrightarrow{n\to+\infty} \mathcal{L}(s) \text{ in } L^p(\mathbb{R})$$

e a meno di una sottosuccessione $\Longrightarrow$ ~~so~~ ho convergenza puntuale quasi ovunque e quindi per quasi ogni $t \in (a, \hat{b})$ ho,

$$\int_{t-\varepsilon_n}^{t} ds\ \varepsilon_n^{-1} \chi'\left(\frac{t-s}{\varepsilon_n}\right) \int_{\mathbb{R}^3} |u(s,x)|^2 \phi(s,x) \longrightarrow$$

$$\int_{\mathbb{R}^3} |u(t,x)|^2\ \phi(t,x)\ dx$$

$$2\int_a^t \int_{\mathbb{R}^3} |\nabla u|^2 \quad \phi(s,x)\, ds\, dx \quad +$$

$$\int_{\mathbb{R}^3} |u(t,x)|^2\ \phi(t,x)\ dx \qquad \not\leq$$

$$\leq \int_a^t \int_{\mathbb{R}^3} |u|^2 \quad \left(\partial_s \phi(s,x) + \Delta\ \phi(s,x) \right)$$

$$+ \int_a^t \int_{\mathbb{R}^3} \left(|u|^2 + 2p \right)\ u \cdot \nabla\ \phi(s,x)$$

Def 1) $u \in L^\infty((a,b), L^2(U))$ $\quad \nabla u \in L^2((a,b)\times U)$

$p \in L^{\frac{3}{2}}((a,b)\times U)$

2) $-\Delta p = \partial_i \partial_j (u_i u_j)$

3) $\forall \quad t \in (a,b)$

$$2\int_a^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi(s,x)\, ds\, dx + \int_{\mathbb{R}^3} |u(t,x)|^2 \phi(t,x)\, dx \le$$

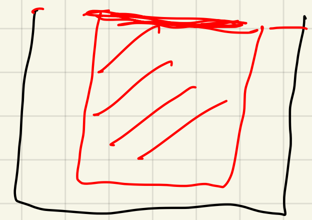
$$\le \int_a^t \int_{\mathbb{R}^3} |u|^2 \left(\partial_s \phi(s,x) + \Delta \phi(s,x) \right)$$

$$+ \int_a^t \int_{\mathbb{R}^3} \left(|u|^2 + 2p \right) u \cdot \nabla \phi(s,x)$$

$$\forall \ \phi \in C_c^\infty\left((a,b) \times U, [0,+\infty) \right)$$

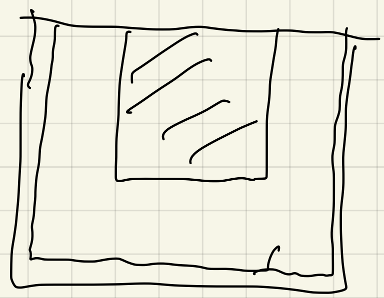
Teor 13.2 Sia (u,p) una suitable pair in $Q_R(t_0,x_0)$. Esistono due costanti $\varepsilon_0^*>0$ e $c_M>0$ t.c. se

$$R^{-2}\int_{Q_R(t_0,x_0)} \left(|u|^3+|p|^{\frac{3}{2}}\right) ds\,dx \le \varepsilon_0$$



dove $\varepsilon_0\in(0,\varepsilon_0^*]$ allora

$$|u|_{L^\infty(Q_{R/2}(t_0,x_0))}\le C_M\,\varepsilon_0^{\frac{1}{3}}.$$



Supponiamo che u soddisfi in $(a,b) \times U$

$\forall\ t \in (a,b)$

$$2\int_a^t \int_{\mathbb{R}^3} |\nabla u|^2 \, \phi(s,x)\, ds\, dx + \int_{\mathbb{R}^3} |u(t,x)|^2 \, \phi(t,x)\, dx \le$$

$$\le \int_a^t \int_{\mathbb{R}^3} |u|^2 \left(\partial_s \phi(s,x) + \Delta \phi(s,x) \right) + \int_a^t \int_{\mathbb{R}^3} |u|^2 \, u \cdot \nabla \phi(s,x)$$

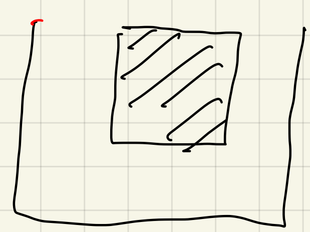
$$\forall\ \phi \in C_c^\infty\left((a,b) \times U, [0,+\infty) \right)$$

Prop. 13.6 Sia u

$Q_R(t_0, x_0)$. Esistono due costanti $\varepsilon_0^* > 0$

e $c_M > 0$ t.c. se

$$R^{-2} \int_{Q_R(t_0,x_0)} |u|^3 \, ds\, dx < \varepsilon_0$$



dove $\varepsilon_0 \in (0, \varepsilon_0^*]$ allora

$$|u|_{L^\infty(Q_{\frac{R}{2}}(t_0,x_0))} \le C_M \, \varepsilon_0^{\frac{1}{3}}$$

$Q_R(t_0, x_0) = Q_1(0,0)$. Si dimostrerà che $\forall (s,a) \in Q_{1/2}(0,0)$ si ha

$$\boxed{\frac{\cancel{2^{5n}} \int_{Q_{2^{-n}}(s,a)} |u|^3 \, dt\, dx}{|Q_{2^{-n}}(s,a)|} \le \epsilon_0^{\frac{2}{3}} \qquad \forall n \in \mathbb{N} \qquad (13.7)}$$

$$|Q_{2^{-n}}| \sim 2^{-5n} = 2^{-2n} \, 2^{-3n}$$

$$Q_{2^{-n}}(0,0) = (-2^{-2n}, 0) \times B_{2^{-n}}(0)$$

$$\phi_n \qquad (\partial_t + \Delta)\phi_n \approx 0$$

$$\phi_n \sim 2^n \qquad \text{in } Q_{2^{-n}}(s,a)$$

$$|\nabla \phi_n| \le \begin{cases} C \; 2^{2n} & \text{in } Q_{2^{-n}}(s,a) \\ C \; 2^{-2n} \, 2^{4k} & \text{in } Q_{2^{-k}}(s,a) \setminus Q_{2^{-(k+1)}}(s,a) \end{cases}$$

$$1 \le k \cancel{\le} n$$

$$2^{2k} \int_{Q_{2^{-k}}(s,a)} |u|^3 \, dt\, dx < \epsilon_0^{\frac{2}{3}} \, 2^{-3k} \qquad \cancel{\forall k}$$

mi serve per $k \le n$

$s - \frac{1}{4} < t < \cancel{0}$ $\quad Q_1(0,0)$

$$2\int_{s-\frac{1}{4}}^{t}\int_{B_{1/2}(a)} |\nabla u|^2 \, \phi_n(s',x)\, ds'\,dx \; + \; \int_{B_{1/2}(a)} |u(t,x)|^2 \, \phi_n(t,x)\, dx \;\lesssim$$

$$\cancel{\le \int_{-1}^{t}\int_{B_{1/2}(a)} |u|^2 \left(\partial_s \phi_n(s,x) + \Delta \phi_n(s,x) \right)}$$

$$+ \int_{s-1/4}^{t}\int_{B_{1/2}(a)} |u|^2 \, u\cdot\nabla \phi_n(s',x)$$

$\phi_n \in C_c^\infty\left(Q_{1/2}(a,s), [0,+\infty)\right)$

$$2^n \int_{B_{2^{-n}}(a)} |u(t,x)|^2 \; + \; 2\, 2^n \int_{Q_{2^{-n}}(a,s)\cap\{t'<t\}} |\nabla u|^2 \, dt'\, dx$$

$$\lesssim \int_{Q_{1/2}(a,s)} |u|^3 \, |\nabla \phi_n| \, dt'\, dx$$

$\phi_n \sim 2^n$ in $Q_{2^{-n}}(s,a)$

$$|\nabla \phi_n| \lesssim \begin{cases} C \; 2^{2n} & \text{in } Q_{2^{-n}}(s,a) \\ C \; 2^{-2n}\, 2^{4k} & \text{in } Q_{2^{-k}}(s,a) \setminus Q_{2^{-(k+1)}}(s,a) \end{cases}$$

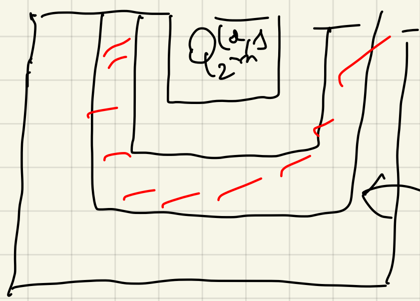
$$2^n \int_{B_{2^{-n}}(a)} |u(t,x)|^2 + 2\, 2^n \int_{Q_{2^{-n}}(a,s) \cap \{t'<t\}} |\nabla u|^2 \, dt'\, dx$$

$$\leq \int_{Q_{\frac{1}{2}}(a,s)} |u|^3 \, |\nabla \phi_n| \, dt'\, dx$$

$$\phi_n \sim 2^n \quad \text{in} \quad Q_{2^{-n}}(s,a)$$

$$|\nabla \phi_n| \lesssim \begin{cases} C \quad 2^{2n} & \text{in } Q_{2^{-n}}(s,a) \\ C \quad 2^{-2n}\, 2^{4k} & \text{in } Q_{2^{-k}}(s,a) \setminus Q_{2^{-(k+1)}}(s,a) \end{cases}$$

$Q_{\frac{1}{2}}(a,s)$



$$Q_{\frac{1}{2}}(a,s) = Q_{2^{-n}}(a,s) \bigcup_{k=1}^{n-1} \left(Q_{2^{-k}}(a,s) \setminus Q_{2^{-(k+1)}}(a,s) \right)$$

$$2^n \int_{B_{2^{-n}}(a)} |u(t,x)|^2 + 2\, 2^n \int_{Q_{2^{-n}}(a,s) \cap \{t'<t\}} |\nabla u|^2 \, dt'\, dx$$

$$\leq \int_{Q_{2^{-n}}(a,s)} |u|^3 \, |\nabla \phi_n| \, dt'\, dx \; +$$

$$+ \sum_{k=1}^{n-1} \int_{Q_{2^{-k}}(a,s) \setminus Q_{2^{-(k+1)}}(a,s)} |u|^3 \, |\nabla \phi_n| \, dt'\, dx$$

$$2^n \int_{B_{2^{-n}}(a)} |u(t,x)|^2 + 2\, 2^n \int_{Q_{2^{-n}}(a,s)\cap\{t'<t\}} |\nabla u|^2 \, dt'\,dx$$

$$\le C 2^{2n} \underbrace{\int_{Q_{2^{-n}}(a,s)} |u|^3 \, dt'\,dx}_{\le \epsilon_0^{\frac{2}{3}} 2^{-5n}} +$$

$$+ 2^{-2n} \sum_{k=1}^{n-1} \underbrace{\int_{Q_{2^{-k}}(a,s)} |u|^3 \, dt'\,dx}_{\le \epsilon_0^{\frac{2}{3}} 2^{-5k}} \; 2^{4k} \le$$

$$\boxed{2^{2k} \int_{Q_{2^{-k}}(s,a)} |u|^3 dt\,dx < \epsilon_0^{\frac{2}{3}} 2^{-3k} \quad \cancel{\forall k} \quad \text{mi vero per } k \le n}$$

$$\le C \epsilon_0^{\frac{2}{3}} 2^{-\cancel{3}n} + C \sum_{k=1}^{n-1} 2^{-k} \epsilon_0^{\frac{2}{3}} 2^{-2n}$$

$$= C 2^{-2n} \epsilon_0^{\frac{2}{3}} \underbrace{\sum_{k=1}^{n} 2^{-k}}$$

$$\frac{1}{2} \sum_{k=0}^{n-1} 2^{-k} < \frac{1}{2} \sum_{k=0}^{\infty} 2^{-k} = 1$$

$$< C\, 2^{-2n} \epsilon_0^{\frac{2}{3}}$$

$$2^n \int_{B_{2^{-n}}(a)} |u(t,x)|^2 \quad + 2\, 2^n \int_{Q_{2^{-n}}(a,s) \cap \{t'<t\}} |\nabla u|^2 \, dt' dx$$

$$< C\, 2^{-2n} \epsilon_0^{\frac{2}{3}}$$

$$2^{n+1} \int_{B_{2^{-n-1}}(a)} |u(t,x)| + 2\, 2^{n+1} \int_{Q_{2^{-n-1}}(a,s) \cap \{t'<t\}} |\nabla u|^2 \, dt' dx$$

$$< 2^3 C \; 2^{-2n-2} \epsilon_0^{\frac{2}{3}}$$

Noi dobbiamo dimostrare

$$2^{2(n+1)} \int_{Q_{2^{-n-1}}(s,a)} |u|^3 \, dt\, dx < \epsilon_0^{\frac{2}{3}} \, 2^{-3(n+1)}$$

<u>Lemma</u> $\exists\, C_0 > 0$ t.c. $\forall\, s \in \mathbb{R}$ e $\forall\, r > 0$ ed $a \in \mathbb{R}$

se $u \in L^\infty((s-r^2, a), L^2(B_r(a)))$ e

se $\nabla u \in L^2(Q_r(a,s))$ allora

$$r^{-2} \int_{Q_r(a,s)} |u|^3 \, dx\, dt \le$$

$$\le C_0 \left[r^{-1} \sup_{t \in [s-r^2, s]} \int_{B_r(a)} |u(t)|^2 dx + r^{-1} \int_{Q_r(a,s)} |\nabla u|^2 dt' dx \right]^{\frac{3}{2}}$$

$$r^{-2}\int_{Q_r(a,s)} |u|^3\,dx\,dt \le$$

$$\le C_0\left[r^{-1}\sup_{t\in[s-r^2,s]}\int_{B_r(a)} |u(t)|^2\,dx + r^{-1}\int_{Q_r(a,s)} |\nabla u|^2\,dt'\,dx\right]^{3/2}$$

$$r = 2^{-n-1}$$

$$2^{2n+2}\int_{Q_{2^{-n-1}}} |u|^3\,dx\,dt' \le$$

$$\le C_0\left[2^{n+1}\sup_{t\in[s-2^{-2n-2},s]}\int_{B_{2^{-n-1}}(a)} |u(t)|^2\,dx + 2^{n+1}\int_{Q_{2^{-n-1}}(a,s)} |\nabla u|^2\,dt'\,dx\right]^{3/2}$$

$$\le C_0\left[2^3 C\, 2^{-\cancel{2}n-2}\,\epsilon_0^{2/3}\right]^{3/2}$$

$$= C_0\left(2^3 C\right)^{3/2} 2^{-3(n+1)}\,\epsilon_0$$

$$< 2^{-3(n+1)}\,\epsilon_0^{2/3}$$

$$C_0\left(2^3 C\right)^{3/2}\epsilon_0^{1/3} < 1$$

$$\phi_n$$

Lemma $\exists\, C_0 > 0$ t.c. $\forall s \in \mathbb{R}$ e $\forall r > 0$ e $\forall a \in \mathbb{R}^3$

se $u \in L^\infty\left((s-r^2, s), L^2(B_r(a))\right)$ e

se $\nabla u \in L^2\left(Q_r(a,s)\right)$ allora

$$r^{-2} \int_{Q_r(a,s)} |u|^3 \, dx\, dt \le$$

$$\le C_0 \left[r^{-1} \sup_{t \in [s-r^2, s]} \int_{B_r(a)} |u(t)|^2 \, dx + r^{-1} \int_{Q_r(a,s)} |\nabla u|^2 \, dt\, dx \right]^{3/2}$$

Dim $r = 1$ $\quad (s, a) = (0, 0)$

$$|u|_{L^3(B_1)} \le |u|^{\frac{1}{2}}_{L^6(B_1)} \, |u|^{\frac{1}{2}}_{L^2(B_1)} \le$$

$$\le c_0 |u|^{\frac{1}{2}}_{L^2(B_1)} \left(|u|^{\frac{1}{2}}_{L^2(B_1)} + |\nabla u|^{\frac{1}{2}}_{L^2(B_1)} \right)$$

$$\le c_0 \left(|u|_{L^2(B_1)} + |u|^{\frac{1}{2}}_{L^2(B_1)} \, |\nabla u|^{\frac{1}{2}}_{L^2(B_1)} \right)$$

$$\int_{B_1} |u|^3 \, dx \le 4 c_0^{3} \left(|u|^3_{L^2(B_1)} + |u|^{\frac{3}{2}}_{L^2(B_1)} \, |\nabla u|^{\frac{3}{2}}_{L^2(B_1)} \right)$$

$$\int_{B_1} |u|^3 dx \le 4 C_0^{\frac{3}{2}} \left(\|u\|^{3}_{L^2(B_1)} + \|u\|^{\frac{3}{2}}_{L^2(B_1)} \|\nabla u\|^{\frac{3}{2}}_{L^2(B_1)} \right)$$

$$\int_{Q} |u|^3 dx \le 4 C_0^{\frac{3}{2}} \int_{-1}^{0} \|u\|^{3}_{L^2(B)} + 4 C_0^{\frac{3}{2}} \int_{-1}^{0} \|u\|^{\frac{3}{2}}_{L^2(B)} \|\nabla u\|^{\frac{3}{2}}_{L^2(B)}$$

$$\le 4 C_0^{\frac{3}{2}} \sup_{-1\le t\le 0} \|u(t)\|^{3}_{L^2(B)} + 4 C_0^{\frac{3}{2}} \sup_{-1\le t\le 0} \|u(t)\|^{\frac{3}{2}}_{L^2(B)} \|\nabla u\|^{\frac{3}{2}}_{L^2_1(Q)}$$

$$\le 4 C_0^{\frac{3}{2}} \sup_{-1\le t\le 0} \|u(t)\|^{3}_{L^2(B)} + 2 C_0^{\frac{3}{2}} \sup_{-1\le t\le 0} \|u(t)\|^{3}_{L^2(B)}$$

$$+ 2 C_0^{\frac{3}{2}} \|\nabla u\|^{3}_{L^2(Q)}$$

$$\int_{B_1} |u|^3 dx \le 6 C_0^{\frac{3}{2}} \left(\sup_{-1\le t\le 0} \|u(t)\|^{2\cdot\frac{3}{2}}_{L^2(B)} + \|\nabla u\|^{2\cdot\frac{3}{2}}_{L^2(Q)} \right)$$

$$\le 6 C_0^{\frac{3}{2}} \left(\sup_{-1\le t\le 0} \|u(t)\|^{2}_{L^2(B)} + \|\nabla u\|^{2}_{L^2(Q)} \right)^{\frac{3}{2}}$$