15 dicembre

$$R(x) = \frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$A = x\,R(x)\Big|_{x=0} = \frac{1}{2}$$

$$B = (x-1)\,R(x)\Big|_{x=1} = -1$$

$$C = (x-2)\,R(x)\Big|_{x=2} = \frac{1}{2}$$

$$\int \frac{1}{x^3-3x^2+2x}\,dx = \int \frac{1}{x(x-1)(x-2)}\,dx =$$

$$= \int \left[\frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2} \right] dx$$

$$= \frac{1}{2} \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \frac{1}{2} \int \frac{1}{x-2}\,dx$$

$$= \frac{1}{2} \lg|x| - \lg|x-1| + \frac{1}{2} \lg|x-2| + C$$

Spieghiamo perchè deve essere in questo caso $A + B + C = 0$

$$\int_3^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx$$

. Questo integrale è convergente, grazie al criterio del confronto. Per $x \in [3, +\infty)$

$$0 < \frac{1}{x(x-1)(x-2)} < \frac{1}{(x-2)^3}$$

$x > x-2 > 0 \iff 0 < \frac{1}{x} < \frac{1}{x-2}$

$x-1 > x-2 > 0 \iff 0 < \frac{1}{x-1} < \frac{1}{x-2}$

$\frac{1}{x-2} = \frac{1}{x-2}$

Verifichiamo che $\dfrac{1}{(x-2)^3} \in L[3,+\infty)$. ✓

$$\lim_{R\to+\infty} \int_3^R \frac{1}{(x-2)^3}\,dx = \lim_{R\to+\infty} \int_1^{R-2} \frac{1}{y^3}\,dy$$

$$y = x-2 \qquad dy = dx$$

$$= \lim_{R\to+\infty} \int_1^R \frac{1}{y^3}\,dy$$

$$= \int_1^{+\infty} \frac{1}{y^3}\,dy \in \mathbb{R}_+$$

$$\Longrightarrow \quad \frac{1}{x(x-1)(x-2)} \in L[3,+\infty)$$

Verifichiamo se ~~Se fosse~~ $\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \quad \cdot x$

deve essere $A+B+C=0$

$$\underbrace{\frac{1}{(x-1)(x-2)}}_{\downarrow\, 0} = A + \underbrace{\frac{x}{x-1}}_{\to 1}B + \underbrace{\frac{x}{x-2}}_{\to 1}C \qquad x \to +\infty$$

$$= A + B + C$$

Piu' in generale, se $R(x) = \dfrac{p(x)}{q(x)}$.

con grado $p \leq$ grado $q - 1$

allora se vale la seguente formula

$$R(x) = \frac{A_1}{x-x_1} + \cdots + \frac{A_N}{x-x_N} \Longrightarrow A_1 + \cdots + A_N = 0$$

$$xR(x) = \frac{x}{x-x_1} A_1 + \cdots + \frac{x}{x-x_N} A_N$$

$$\lim_{x\to+\infty} x R(x) = \lim_{x\to+\infty} \frac{x\, p(x)}{q(x)} = 0 \,, \quad \overbrace{\lim_{x\to+\infty} \left(\frac{x}{x-x_1} A_1 + \cdots + \frac{x}{x-x_N} A_N \right)}^{A_1 + \cdots + A_N \, =}$$

Calcoliamo esplicitamente $\displaystyle\int_3^{+\infty} \frac{1}{x^3-3x^2+x}\,dx = \int_3^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx$

$$\lim_{R\to+\infty} \int_3^R \frac{dx}{x(x-1)(x-2)}$$

$$\int_3^R \frac{dx}{x(x-1)(x-1)} = \left(\frac{1}{2}\lg x - \lg(x-1) + \frac{1}{2}\lg(x-2)\right)\Bigg]_3^R$$

$$= \lg \frac{\sqrt{x^2-2x}}{x-1}\Bigg]_3^R = \lg\left(\frac{\sqrt{R^2-2R}}{R-1}\right) - \lg\left(\frac{\sqrt{3}}{2}\right) \xrightarrow{R\to+\infty}$$

$$\int_3^{+\infty} \frac{1}{x^3-3x^2+x}\,dx = -\lg\left(\frac{\sqrt{3}}{2}\right) > 0$$

$\frac{\sqrt{3}}{2} < 1 \qquad \sqrt{3} < 2$

$3 < 4$

Teor Sia $R(z) = \frac{P(z)}{Q(z)}$ $P(z)$ e $Q(z)$ polinomi con grado $P(z) <$ grado $Q(z) = n \geq 1$ e sia $Q(z)$ della forma

$$Q(z) = a_n (z-z_1)^{m_1} \dots (z-z_k)^{m_k} \qquad m_1 + \dots + m_k = n$$

Allora esistono delle costanti

z_1 $\quad A_{11} \dots A_{1m_1}$

$\vdots$

z_k $\quad A_{k1} \dots A_{k m_k}$

$$R(z) = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \frac{A_{j\ell}}{(z-z_j)^\ell}$$

$$\begin{array}{llll} z_1 & A_{11} & \cdots & A_{1m_1} \\ \vdots & & & \\ z_k & A_{k1} & & A_{km_k} \end{array}$$

$$R(z) = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \frac{A_{j\ell}}{(z-z_j)^\ell}$$

$$A_{j\ell} = \frac{1}{(m_j-\ell)!} \left(\frac{d}{dz}\right)^{m_j-\ell} \left(R(z)(z-z_j)^{m_j} \right) \Big|_{z=z_j}$$

$$B = R(x)\,x^2\Big|_{x=0} = \frac{1}{x-1}\Big|_{x=0} = -1$$

$$A = \frac{d}{dx}\left(R(x)\,x^2\right)\Big|_{x=0} = \frac{d}{dx}\frac{1}{x-1}\Big|_{x=0} = \frac{-1}{(x-1)^2}\Big|_{x=0} = -1$$

$$R(x) = \left(\frac{A}{x} + \frac{B}{x^2}\right) + \frac{C}{x-1} = \frac{1}{x^2(x-1)}$$

$$A_{jl} = \frac{1}{(m_j-l)!}\left(\frac{d}{dz}\right)^{m_j-l}\left(R(z)\,(z-z_j)^{m_j}\right)\Big|_{z=z_j}$$

$$C = R(x)\,(x-1)\Big|_{x=1} = \frac{1}{x^2}\Big|_{x=1} = 1$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{1}{x^2(x-1)} \quad , x$$

$$A + \underbrace{\frac{B}{x}}_{\to 0} + \frac{C\,x}{x-1} = \underbrace{\frac{1}{x(x-1)}}_{\to 0} \qquad x \longrightarrow +\infty$$

$$A + C = 0$$

$$R(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z^2+1)(z^2+1)} = \frac{1}{(z-i)(z+i)(z-i)(z+i)}$$

$$= \frac{1}{(z-i)^2(z+i)^2} = \frac{A}{z-i} + \frac{B}{z+i} + \frac{C}{(z-i)^2} + \frac{D}{(z+i)^2}$$

$$A_{j\ell} = \frac{1}{(m_j-\ell)!}\left(\frac{d}{dz}\right)^{m_j-\ell}\left(R(z)(z-z_j)^{m_j}\right)\Big|_{z=z_j}$$

$$D = R(z)(z+i)^2\Big|_{z=-i} = \frac{1}{(z-i)^2}\Big|_{z=-i} = \frac{1}{(-2i)^2} = -\frac{1}{4}$$

$$C = R(z)(z-i)^2\Big|_{z=i} = \frac{1}{(z+i)^2}\Big|_{z=i} = \frac{1}{(2i)^2} = -\frac{1}{4}$$

$$A_{jl} = \frac{1}{(m_j - l)!} \left(\frac{d}{dz}\right)^{m_j - l} \left(R(z)(z-z_j)^{m_j} \right) \Big|_{z=z_j}$$

$$A = \frac{d}{dz} \left(R(z)(z-i)^2 \right) \Big|_{z=i} = \frac{d}{dz} \frac{1}{(z+i)^2} \Big|_{z=i} = \frac{-2}{(z+i)^3} \Big|_{z=i}$$

$$= \frac{-2}{(2i)^3} = \frac{-2}{2^3(-i)} = \frac{1}{4i} = \frac{-i}{4}$$

$$B = \frac{d}{dz} \left(R(z)(z+i)^2 \right) \Big|_{z=-i} = \frac{d}{dz} \frac{1}{(z-i)^2} \Big|_{z=-i} = \frac{-2}{(z-i)^3} \Big|_{z=-i}$$

$$= \frac{-2}{(-2i)^3} = \frac{i}{4}$$

$$\frac{1}{(z^2+1)^2} = -\frac{i}{4}\ \frac{1}{z-i} + \frac{i}{4}\ \frac{1}{z+i} - \frac{1}{4}\ \frac{1}{(z-i)^2} - \frac{1}{4}\ \frac{1}{(z+i)^2}$$

$$\frac{1}{(x^2+1)^2} = -\frac{i}{4}\ \frac{1}{x-i} + \frac{i}{4}\ \frac{1}{x+i} - \frac{1}{4}\ \frac{1}{(x-i)^2} - \frac{1}{4}\ \frac{1}{(x+i)^2}$$

$$= \frac{i}{4}\left(\frac{1}{x+i} - \frac{1}{x-i}\right) + \frac{1}{4}\ \frac{d}{dx}\ \frac{1}{(x-i)} + \frac{1}{4}\ \frac{d}{dx}\ \frac{1}{x+i}$$

$$= \frac{i}{4}\ \frac{\cancel{x}-i-(\cancel{x}+i)}{x^2+1} + \frac{1}{4}\ \frac{d}{dx}\left[\frac{1}{x-i} + \frac{1}{x+i}\right]$$

$$= \frac{i}{4}\ \frac{(-2i)}{x^2+1} + \frac{1}{4}\ \frac{d}{dx}\ \frac{x+\cancel{i}+x-\cancel{i}}{x^2+1} = \frac{1}{2}\ \frac{1}{x^2+1} + \frac{1}{2}\ \frac{d}{dx}\ \frac{x}{x^2+1}$$