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FOGLIO 9

① $A(x) = \begin{pmatrix} x-1 & x^2 & 1 \\ 0 & -2x & 3 \\ 1 & x & 0 \end{pmatrix}$.

RANGO E DET IN $\mathbb{R}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$?

• DET? $\begin{vmatrix} x-1 & x^2 & 1 \\ 0 & -2x & 3 \\ 1 & x & 0 \end{vmatrix} \underset{R_1 \to R_1+(-1)R_3}{=} \begin{vmatrix} 0 & x & 1 \\ 0 & -2x & 3 \\ 1 & x & 0 \end{vmatrix} \underset{R_2 \to R_2+2R_1}{\overset{\circledast}{=}} \begin{vmatrix} 0 & x & 1 \\ 0 & 0 & 5 \\ 1 & x & 0 \end{vmatrix} \underset{\text{sviluppo prima colonna}}{=}$

$- \begin{vmatrix} x & 1 \\ 0 & 5 \end{vmatrix} = -5x. \Rightarrow$ IN $\mathbb{Z}_5 = 0$, ALTRIMENTI $= 0 \Leftrightarrow x=0$.

• RANGO?

OSSERVIAMO LA MATRICE RIDOTTA DOPO IL PASSAGGIO $\circledast$ (↑).

• IN $\mathbb{Z}_5$: $\begin{pmatrix} 0 & x & 1 \\ 0 & 0 & 5 \\ 1 & x & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 1 \\ 0 & 0 & 0 \\ 1 & x & 0 \end{pmatrix} \Rightarrow rg(A(x)) = 2 \ \forall x \in \mathbb{Z}_5$

• IN $\mathbb{R}, \mathbb{Z}_2, \mathbb{Z}_3$: $rg \begin{pmatrix} 0 & x & 1 \\ 0 & 0 & 5 \\ 1 & x & 0 \end{pmatrix} = \begin{cases} 3 & x \neq 0 \\ 2 & x = 0 \end{cases}$.

$B(x) = \begin{pmatrix} 3-x & 1 & -1 \\ 2 & x-1 & 0 \\ 0 & -1 & x \end{pmatrix}$

• DET?

$\begin{vmatrix} 3-x & 1 & -1 \\ 2 & x-1 & 0 \\ 0 & -1 & x \end{vmatrix} \underset{R_3 \to R_3 + xR_1}{=} \begin{vmatrix} 3-x & 1 & -1 \\ 2 & x-1 & 0 \\ 3x-x^2 & x-1 & 0 \end{vmatrix} \underset{R_3 \to R_3 - R_2}{\overset{\circledast}{=}} \begin{vmatrix} 3-x & 1 & -1 \\ 2 & x-1 & 0 \\ -x^2+3x-2 & 0 & 0 \end{vmatrix} =$

$= - \begin{vmatrix} 2 & x-1 \\ -(x-2)(x-1) & 0 \end{vmatrix} = -(x-1)^2 (x-2)$

$\Rightarrow \det B(x) = 0 \iff (x-2 = 0 \ \vee \ (x-1) = 0)$

$\rightarrow$ in $\mathbb{R}, \mathbb{Z}_3, \mathbb{Z}_5$: $\det B(x) = 0 \iff x=1, x=2$.

in $\mathbb{Z}_2$: $\det B(x) = 0 \iff x=1, x=2=0 \rightarrow \forall x$.

RANGO?

Sappiamo che per $x \neq 1$, $x \neq 2$ $\det(B(x)) \neq 0 \rightarrow rg = 3$.

DAL PASSAGGIO $\circledast$, PER TUTTI I CAMPI TRANNE $\mathbb{Z}_2$:

$\begin{pmatrix} 3-x & 1 & -1 \\ 2 & x-1 & 0 \\ -(x-1)(x-2) & 0 & 0 \end{pmatrix} \rightarrow rg \geq 2 \ \forall x \rightarrow \begin{cases} rg = 2 \text{ se } x=1, x=2 \\ rg = 3 \text{ se } x \neq 1, 2. \end{cases}$

↑ prime due righe sono l.i.

IN $\mathbb{Z}_2$

$\begin{pmatrix} 3-x & 1 & -1 \\ 2 & x-1 & 0 \\ -(x-2)(x-1) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+x & 1 & 1 \\ 0 & 1+x & 0 \\ x(x+1) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+x & 1 & 1 \\ 0 & 1+x & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow rg(B(x)) = \begin{cases} 2 & \text{se } x+1 \neq 0 \quad (x=0) \\ 1 & \text{se } x+1 = 0 \quad (x=1) \end{cases}$ ✓✓.

②

$V_1 = (2\ 0\ 1\ 0)$, $V_2 = (0\ 3\ 1\ 1)$

$\rightarrow \left\{ V_1, V_2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ base di $\mathbb{R}^4$.

$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2 \cdot 3 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 2 \cdot 3 \cdot 1 = 6$

$\rightarrow$ CONSIDERIAMO $\left\{ V_1, V_2, \begin{pmatrix} 0 \\ 0 \\ 1/6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ base di $\mathbb{R}^4$.

$\rightarrow \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 1/6 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 3 & 0 & 0 \\ 1 & 1/6 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2 \cdot 3 \cdot \begin{vmatrix} 1/6 & 0 \\ 0 & 1 \end{vmatrix} = 6 \cdot \frac{1}{6} = 1$ ✓✓.

③ RIPRENDIAMO $A(x) = \begin{pmatrix} x-1 & x^2 & 1 \\ 0 & -2x & 3 \\ 1 & x & 0 \end{pmatrix}$, $B(x) = \begin{pmatrix} 3-x & 1 & -1 \\ 2 & x-1 & 0 \\ 0 & -1 & x \end{pmatrix}$

$\operatorname{Ker} L_{A(x)}$?

Sappiamo $rg(A(x)) = 3 \ \forall x \neq 0$

$\rightarrow$ per $x \neq 0$, $\operatorname{Ker} L_{A(x)} = \{\underline{0}\}$.

$x = 0$?

$A(0) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow A(0) \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -y_1 + y_3 = 0 \\ 3y_3 = 0 \\ y_1 = 0 \end{cases}$

$\Rightarrow \operatorname{Ker} L_{A(0)} = \{ (0, y_2, 0) \in \mathbb{R}^3 \mid y_2 \in \mathbb{R} \} = \langle (0,1,0) \rangle$ ✓.

$\operatorname{Ker} L_{B(x)}$?

Sappiamo $rg(B(x)) = 3 \ \forall x \neq 1, 2$. $\rightarrow \forall x \neq 1, 2 \ \operatorname{Ker} L_{B(x)} = \{\underline{0}\}$.

$x = 1$

$B(1) = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{cases} 2y_1 + y_2 - y_3 = 0 \\ 2y_1 = 0 \\ -y_2 + y_3 = 0 \end{cases} \rightarrow \begin{cases} y_1 = 0 \\ y_2 = y_3 \end{cases}$

$\Rightarrow \operatorname{Ker} L_{B(x)} = \{ (0, y_3, y_3) \mid y_3 \in \mathbb{R} \} = \langle (0,1,1) \rangle$ ✓✓.