18 dicembre

$$\int_1^{\infty} \lg\left(1+\frac{1}{x^p}\right)dx$$ per $p>0$, stabilire quando è sommabile.

Osserviamo che $\displaystyle\lim_{x\to+\infty} \frac{\lg\left(1+\frac{1}{x^p}\right)}{\frac{1}{x^p}} =$

$y=\frac{1}{x^p}$

$$= \lim_{y\to 0} \frac{\lg(1+y)}{y} = 1 .$$

$$\log\left(1+\frac{1}{x^p}\right) \in L[1,+\infty) \iff \frac{1}{x^p} \in L[1,+\infty)$$

Sappiamo che $\frac{1}{x^p} \in L[1,+\infty) \iff p > 1.$

Quindi

$$\log\left(1+\frac{1}{x^p}\right) \in L[1,+\infty) \iff p>1.$$

Sommabilità di $\displaystyle\int_1^{+\infty} \frac{1}{[x]^p}\,dx$

Confrontiamo $\dfrac{1}{[x]^p}$ con $\dfrac{1}{x^p}$

Osserviamo che per $x \to +\infty$, si ha

$$[x] = x\,(1 + o(1))$$

$$[x] = x - x + [x] = x - (x - [x]) = x\left(1 - \frac{x - [x]}{x}\right)$$

$$[x] = x\,(1 + o(1)) \quad \text{per } x \to +\infty \qquad (*)$$

$$[x] = x\left(1 - \frac{x - [x]}{x}\right)$$

Verifichiamo ora che $\displaystyle\lim_{x \to +\infty} \frac{x - [x]}{x} = 0$

Ricordiamo che $\forall x$, $\quad [x] \le x < [x] + 1 \qquad -[x]$

$$0 \le x - [x] < 1 \qquad \cdot \frac{1}{x} \qquad x > 1$$

$$\underset{\downarrow\,0}{0} \le \frac{x - [x]}{x} < \underset{\downarrow\,0}{\frac{1}{x}} \quad x \to +\infty \quad \Rightarrow \lim_{x \to +\infty} \frac{x - [x]}{x} = 0$$

$$\lim_{x\to+\infty} \frac{\frac{1}{[x]^p}}{\frac{1}{x^p}} = \lim_{x\to+\infty} \frac{x^p}{[x]^p} =$$

$$= \lim_{x\to+\infty} \frac{x^p}{\left(x\,(1+o(1))\right)^p} = \lim_{x\to+\infty} \frac{\cancel{x^p}}{\cancel{x^p}\,(1+o(1))^p}$$

$$= 1 \qquad\qquad \frac{1}{(1+o(1))^p} = 1+o(1)$$

$$\frac{1}{[x]^p} \in L[1,+\infty) \iff \frac{1}{x^p} \in L[1,+\infty) \iff p>1.$$

$$\int_0^{+\infty} 2^{-[3x]}\,dx \qquad \text{Calcolarlo.}$$

E' integrabile?

$$[3x] \leq 3x < [3x] + 1$$

$$-[3x] \geq -3x > -[3x] - 1 \implies -[3x] < -3x + 1$$

$$0 < 2^{-[3x]} < 2^{-3x} \cdot 2 \qquad \text{e risulta} \quad 2^{-3x} \in L[0,+\infty)$$

$$\int_0^y 2^{-3x}\,dx = \int_0^y e^{\lg 2^{-3x}}\,dx = \int_0^y e^{-3x\lg 2}\,dx =$$

$$= \left.\frac{e^{-3x\lg 2}}{-3\lg 2}\right]_0^y = \frac{1}{3\lg 2} - \frac{e^{-3y\lg 2}}{3\lg 2} \xrightarrow[y\to+\infty]{} \frac{1}{3\lg 2}$$

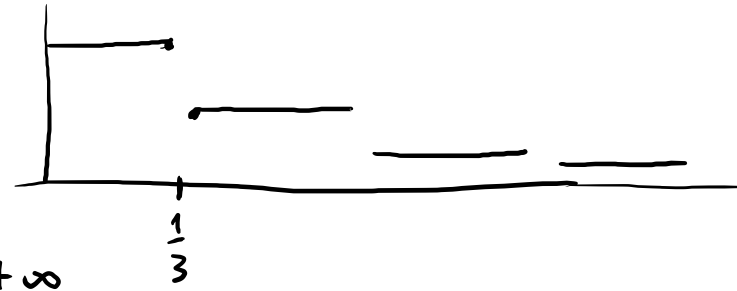
$0 < 2^{-[3x]} < 2 \; 2^{-3x} \implies$ per confronto $2^{-[3x]} \in L([0,+\infty))$

Esercizio calcolare $\displaystyle\lim_{x\to+\infty} \frac{2^{-[3x]}}{2^{-3x}}$

$$\int_0^{+\infty} 2^{-[3x]}\,dx$$

$$y = 3x \qquad dy = 3\,dx$$

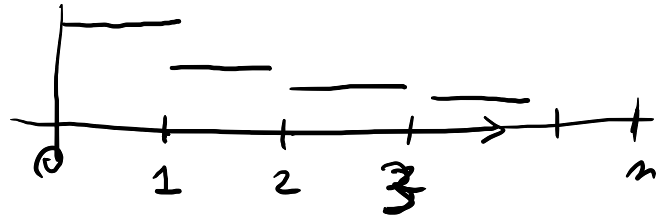
$$= \frac{1}{3}\int_0^{+\infty} 2^{-[y]}\,dy = \frac{1}{3}\int_0^{+\infty} 2^{-[x]}\,dx$$



$$\cancel{\int_0^{+\infty} e^{-[x]}\,dx = -e^{-[x]}\Big]_0^{+\infty} = e^{[0]} = e^{0} = 1}$$

$$\int_0^{+\infty} 2^{-[x]}\,dx = \lim_{y\to+\infty}\int_0^y 2^{-[x]}\,dx =$$

$$= \lim_{n\to+\infty}\int_0^n 2^{-[x]}\,dx$$



$$= \lim_{n\to+\infty}\sum_{j=1}^{n}\int_{j-1}^{j} 2^{-[x]}\,dx$$

$$= \lim_{n\to+\infty}\sum_{j=1}^{n}\int_{j-1}^{j} 2^{-(j-1)}\,dx = \lim_{n\to+\infty}\sum_{j=1}^{n} 2^{-(j-1)} \qquad k=j-1$$

$$= \lim_{n\to+\infty}\sum_{k=0}^{n-1} 2^{-k} = \lim_{n\to+\infty}\frac{1-2^{-n}}{1-2^{-1}} = \frac{1}{1-2^{-1}} = 2$$

Def Sia $f \in L_{loc}[a,b)$ $\quad a \in \mathbb{R}$ e $b \in \mathbb{R} \cup \{+\infty\}$.
Diciamo che f è assolutamente integrabile in $[a,b)$ se $\quad |f| \in L[a,b)$.

Teor Sia $f \in L_{loc}[a,b)$. Se f è assolutamente integrabile in $[a,b)$, allora $f \in [a,b)$.

__Es__ Sia $f(x) = \begin{cases} 1 & \text{se } x \in \mathbb{Q} \\ -1 & \text{se } x \notin \mathbb{Q} \end{cases}$

allora $\forall\ a < b$ in $\mathbb{R}$, $|f(x)| \in L([a,b])$

mentre $f \notin L([a,b])$

$$|f(x)| \equiv 1 \implies |f(x)| \in L([a,b])$$

$$f(x) = 2D(x) - 1 \qquad D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$f \notin L[a,b]$ $\forall\ [a,b]$, perché se esistesse

Un intervallo $[a,b]$ t.c. $f \in L[a,b]$ allora,

da $f(x) = 2D(x) - 1 \Longrightarrow D(x) = \frac{f(x)+1}{2}$

$\Longrightarrow D \in L[a,b]$ assurdo.

Stabilire se $\sin(x)\, e^{-x}\, th(x) \in L[0,\infty)$

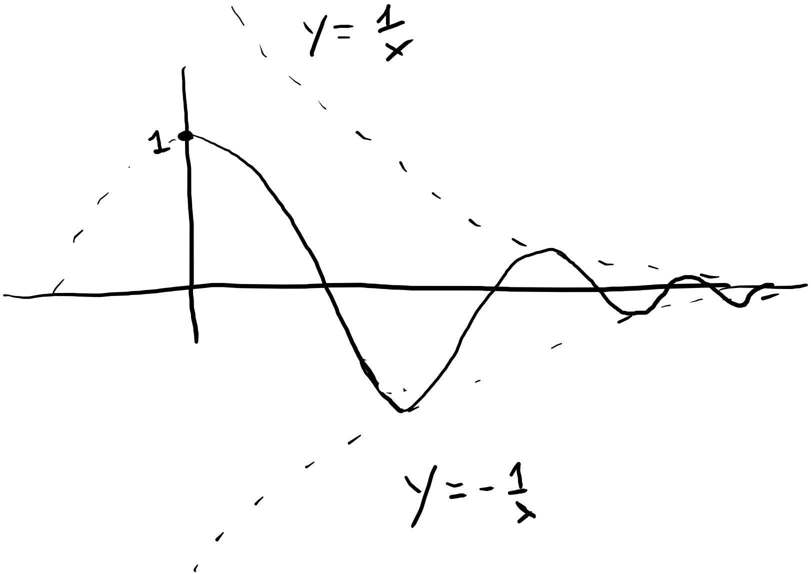
$$|\sin(x)\, e^{-x}\, th(x)| \leq |\sin x|\, e^{-x}\, th(x)$$

$$\leq e^{-x} \in L[0,+\infty)$$

$\Longrightarrow$ per confronto $|\sin(x)\, e^{-x}\, th(x)| \in L[0,\infty)$

$\Longrightarrow$ $\sin(x)\, e^{-x}\, th(x) \in L[0,\infty)$.

$f(x) = \frac{\sin x}{x} \in L[0,+\infty)$ $\qquad$ $|f(x)| = \frac{|\sin x|}{x} \notin L[0,\infty)$



Dimostriamo che $f \in L[0,+\infty) \iff f \in L[1,+\infty)$

$$\int_1^Y f(x)\,dx = \int_1^Y \frac{\sin x}{x}\,dx = \int_1^Y \frac{1}{x}\,(-\cos x)'\,dx$$

$$= -\int_1^Y \frac{1}{x}(\cos x)'\,dx = -\left(\left.\frac{\cos x}{x}\right]_1^Y + \int_1^Y \frac{1}{x^2}\cos x\,dx\right)$$

$$= \cos 1 - \frac{\cos Y}{Y} - \int_1^Y \frac{\cos x}{x^2}\,dx$$

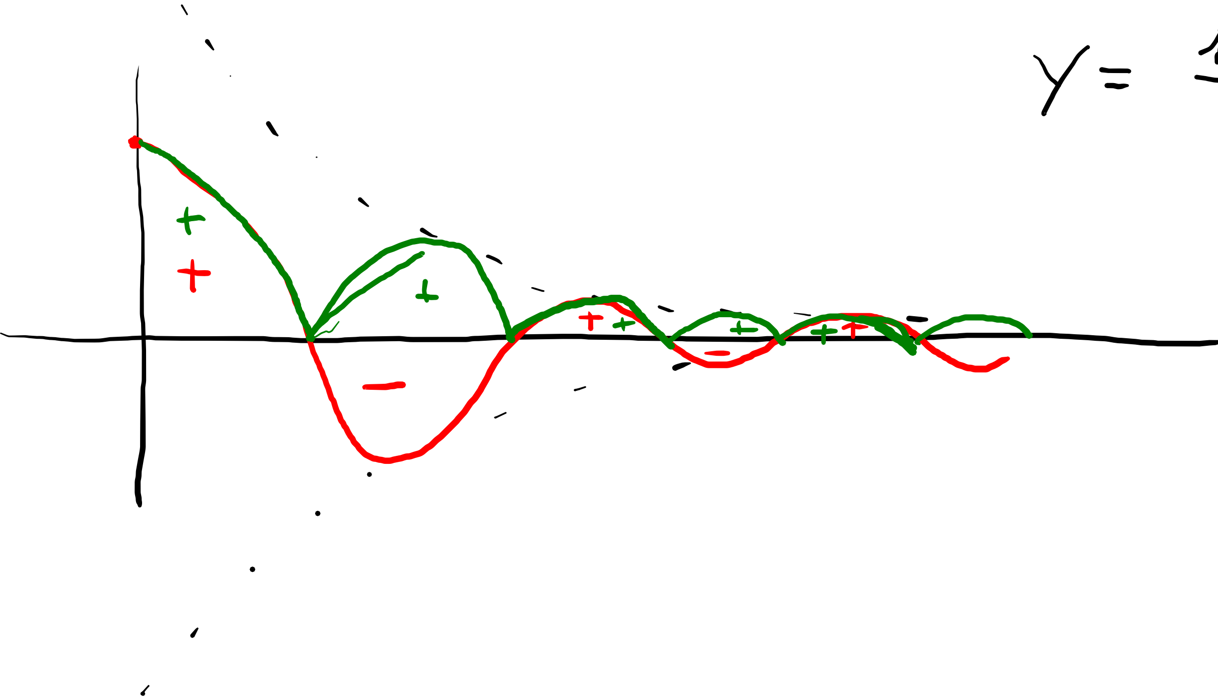
$$\lim_{Y\to+\infty}\left(\cos 1 - \frac{\cos Y}{Y} - \int_1^Y \frac{\cos x}{x^2}\,dx\right) = \cos 1 - \lim_{Y\to+\infty}\int_1^Y \frac{\cos x}{x^2}\,dx$$

$\frac{\cos x}{x^2} \in L[1,+\infty)$ perché $\frac{|\cos x|}{x^2} \in L[1,+\infty)$

e quest'ultimo fatto è vero perché

$$0 \leq \frac{|\cos x|}{x^2} \leq \frac{1}{x^2} \in L[1,+\infty)$$

Dimostriamo che $\frac{|\operatorname{sen} x|}{x} \notin L[0,+\infty)$

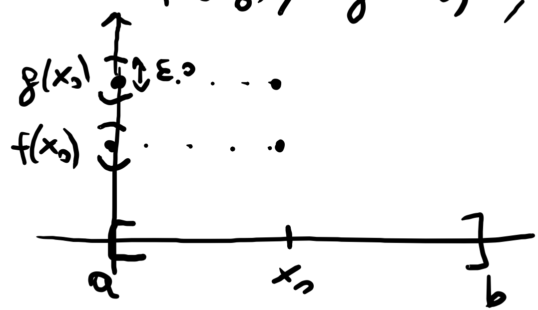


$$y = \frac{\operatorname{sen} x}{x}$$

Esercizio Siano $f, g \in C^0([a,b])$ $\quad f(x) \le g(x)$ $\forall x$ e non la stessa funzione. Allora $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$.

Dim Se non è vero che $f(x) = g(x)$ $\forall x \in [a,b]$ allora $\exists$ $x_0 \in [a,b]$ t.c. $f(x_0) \neq g(x_0)$, cioè

$$f(x_0) < g(x_0)$$



Se io prendo

$$0 < \varepsilon_0 < \frac{f(x_0) - g(x_0)}{2}$$

Segue che $\quad f(x_0) < f(x_0) + \varepsilon_0 < g(x_0) - \varepsilon < g(x_0)$

Dalla continuità di f e g in ε_0 so so che

$\exists\, \delta_0 > 0$ t.c.

$0 \leq |x - x_0| \leq \delta_0 \Longrightarrow$ so che $|f(x) - f(x_0)| < \varepsilon_0$

$x \in [a, b]$ che $|g(x) - g(x_0)| < \varepsilon_0$

cioè anche

$$f(x) < f(x_0) + \varepsilon_0 < g(x_0) - \varepsilon_0 < g(x)$$

$\forall x \in [a, b] \cap [x_0 - \delta_0, x_0 + \delta_0]$

$$\int_{x_0-\delta_0}^{x_0+\delta_0} f(x)\,dx < 2\delta_0\left(f(x_0)+\varepsilon_0\right) < 2\delta_0\left(g(x_0)-\varepsilon_0\right) <$$

$$< \int_{x_0-\delta_0}^{x_0+\delta_0} g(x)\,dx \qquad [x_0-\delta_0,\, x_0+\delta_0] \subseteq [a,b].$$

$$\int_a^b f(x)\,dx = \underbrace{\int_a^{x_0-\delta_0} f(x)\,dx}_{\le \int_a^{x_0-\delta} g} + \underbrace{\int_{x_0-\delta_0}^{x_0+\delta_0} f(x)\,dx}_{< \int_{x_0-\delta_0}^{x_0+\delta_0} g} + \underbrace{\int_{x_0+\delta_0}^{b} f(x)\,dx}_{\le \int_{x_0+\delta_0}^{b} g}$$