$$\begin{cases} x'(t) = a\,x(t) + b\,y(t) \\ y'(t) = c\,x(t) + d\,y(t) \end{cases}$$

a, b, c, d costanti reali

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\operatorname{tr} A = a + d$$

$$\det A = ad - bc$$

$$\boxed{X' = A \cdot X}$$

- $z'' - (a+d)\,z' + (ad - bc)\,z = 0$

$$z'' = -h z' - k z$$

$$k = cd - bz$$

$$h = -(a+d)$$

$$\underbrace{z'' - (a+d)z'}_{h} + \underbrace{(ad-bc)\,z}_{k} = 0$$

eq differenziale (normale, autonomo del secondo ordine)

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \iff \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

associato al sistema di eq. differenziali del primo ordine, lineari, autonomi e a coefficienti costanti

$$k = \det A \quad \operatorname{tr} A = -h$$

$$\Delta = h^2 - 4k = (a+d)^2 - 4(ad-bc) =$$

$$= a^2 + 2ad + d^2 - 4ad + 4bc =$$

$$= \underbrace{a^2 - 2ad + d^2}_{(a-d)^2} + 4bc = (a-d)^2 + 4bc$$

$\Delta > 0 \qquad z(t) = A \cdot e^{-m_1 t} + B \cdot e^{-m_2 t}$

$${}_1m_2 = \frac{h \mp \sqrt{\Delta}}{2}$$

$\Delta = 0 \qquad \underset{\substack{\| \\ m}}{m_1} = m_2 = h/2 \qquad z(t) = (A + Bt) \cdot e^{-mt}$

$\Delta < 0 \qquad p = h/2 \qquad q = \frac{\sqrt{-\Delta}}{2}$

$$z(t) = e^{-pt} \cdot (A\cos qt + B\operatorname{sen} qt)$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ matrice quadrata 2×2
con a, b, c, d numeri reali

$$\begin{pmatrix} \boxed{a \quad b} \\ \boxed{c \quad d} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: \begin{pmatrix} \underline{ax+by} \\ \underline{cx+dy} \end{pmatrix}$$

<u>Teorema</u> Data una matrice 2×2 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ con a, b, c, d reali, allora è possibile trovare una base di $\mathbb{R}^2$ rispetto alla quale la matrice A si scrive

① $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ se $\underline{\Delta > 0}$ ② $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ o $\begin{pmatrix} m & 1 \\ 0 & m \end{pmatrix}$ se $\underline{\Delta = 0}$

③ $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ se $\Delta < 0$ $\quad p = h/2 \quad q = \dfrac{\sqrt{-\Delta}}{2}$

Caso ① $\quad \Delta > 0 \quad h^2 - 4k > 0 \quad m_{1,2} = \dfrac{-(a+d) \mp \sqrt{(a-d)^2 + 4bc}}{2}$

$h = -(a+d)$

$k = ad - bc$

$x_1 := cx + (m_1 - a) y$

$y_1 := cx + (m_2 - a) y$

$\rightsquigarrow \begin{cases} x_1' = m_1 x_1 \\ y_1' = m_2 \cdot y_1 \end{cases}$ DISACCOPPIATE

$\boxed{x_1 = A e^{m_1 t} \quad y_1 = B e^{m_2 t}}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\substack{\shortparallel \\ A}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x_1 = cx + (m_1 - a)y$$
$$y_1 = cx + (m_2 - a)y$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \underbrace{\begin{pmatrix} c & m_1 - a \\ c & m_2 - a \end{pmatrix}}_{B_1} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$x_1' = m_1 x_1$$
$$y_1' = m_2 \cancel{x} y_1$$

$$\boxed{\begin{aligned} x_1 &= A e^{m_1 t} \\ y_1 &= B e^{m_2 t} \end{aligned}}$$

$$B_1^{-1} = \begin{pmatrix} \dfrac{m_2 - a}{c\,(m_2 - m_1)} & \dfrac{a - m_1}{c\,(m_2 - m_1)} \\[2ex] \dfrac{-1}{m_2 - m_1} & \dfrac{1}{m_2 - m_1} \end{pmatrix}$$

$$B_1 \cdot B_2^{-1} = B_1^{-1} \cdot B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{\begin{pmatrix} x \\ y \end{pmatrix}} = B_1^{-1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = B_1^{-1} \begin{pmatrix} Ae^{m_1 t} \\ Be^{m_2 t} \end{pmatrix}$$

x – —

y · —

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{con} \quad \det(A) = \underline{ad - bc \neq 0}$$

$$A^{-1} = \begin{pmatrix} +\dfrac{d}{\det A} & -\dfrac{b}{\det A} \\ -\dfrac{c}{\det A} & \dfrac{a}{\det A} \end{pmatrix}$$

2 $\Delta = 0$ ne $b = c = 0$ $\qquad \Delta = (a-d)^2 + 4bc$

$\Rightarrow a = d$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{matrix} Ae^{at} = x(t) \\ Be^{at} = y(t) \end{matrix}$$

$b \neq 0$

$$\begin{cases} x_1 := \dfrac{x}{b} \\ y_1 = \dfrac{a-d}{2b} x + y \end{cases} \iff \boxed{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{b} & 0 \\ \frac{a-d}{2b} & 1 \end{pmatrix}}_{= B_2} \begin{pmatrix} x \\ y \end{pmatrix}}$$

$$x_1 = \frac{x}{b}$$

$$y_1 = \frac{a-d}{2b}\, x + y$$

$$\begin{cases} x_1' = m x_1 + y_1 \\ y_1' = m y_1 \end{cases}$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} m & 1 \\ 0 & m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{y_1 = A \cdot e^{mt}}$$

$$\boxed{x_1'(t) = m\, x_1(t) + A e^{mt}}$$

eq. diff.
LINEARE
nella sola
variabile
x_1

Utilizzando la formula per trovare una soluzione di una eq. lineare del primo ordine

$$x' = B(t)\,x + C(t)$$

$$B(t) = m \qquad \mathcal{B}(t) = mt$$

$$C(t) = A e^{mt}$$

$$x_1(t) = \left(\int A \underbrace{e^{+mt} \cdot e^{+mt}} \, dt + \mathcal{B} \right) e^{+mt}$$

$$= \left(A t + \mathcal{B} \right) e^{mt}$$

$$B_2 = \begin{pmatrix} \frac{1}{b} & 0 \\ \frac{a-d}{2b} & 1 \end{pmatrix} \qquad \det B_2 = \frac{1}{b} \neq 0$$

$$B_2^{-1} = \begin{pmatrix} b & 0 \\ \frac{d-a}{2} & 1 \end{pmatrix}$$

3

$\Delta < 0$ $\qquad p = \frac{h}{2} \qquad q = \frac{\sqrt{-\Delta}}{2}$

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \rightsquigarrow \begin{cases} x_1' = px_1 + qy_1 \\ y_1' = qx_1 + py_1 \end{cases}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

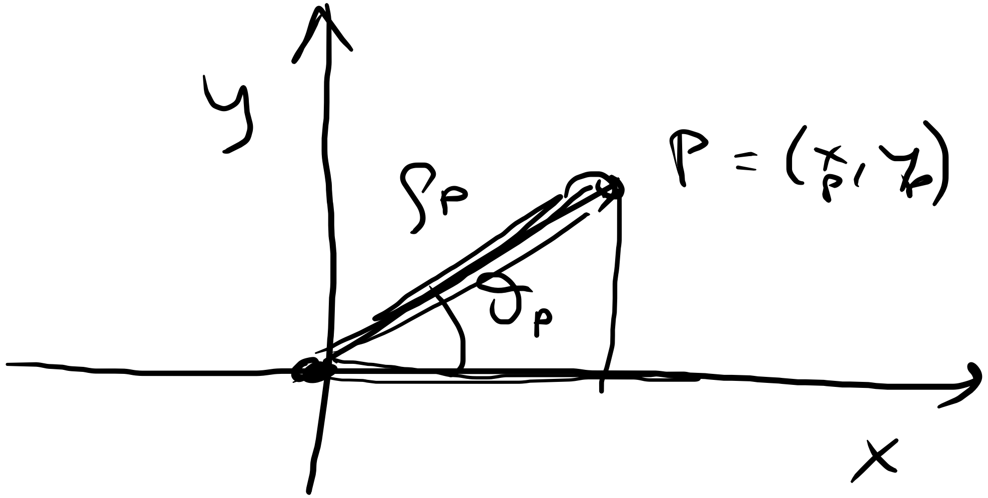
$c \neq 0$

$$\begin{cases} x_1 = cx + (p-a)y \\ y_1 = qy \end{cases} \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \underbrace{\begin{pmatrix} c & p-a \\ 0 & q \end{pmatrix}}_{= B_3} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} x_1' = p x_1 - q y_1 \\ y_1' = q x_1 + p y_1 \end{cases}$$

non disaccoppiata

NOTAZIONI in <u>coordinate polari</u>



$P = O$

$$P = (\overset{x_P}{\overset{\shortparallel}{\rho_P \cdot \cos\vartheta_P}}, \overset{y_P}{\rho_P \cdot \operatorname{sen}\vartheta_P})$$

ρ_P intensità o modulo di $\overline{OP}$

ϑ_P argomento

$$\rho_p = \sqrt{x_p^2 + y_p^2}$$

$$x_p = \rho_p \cdot \cos\theta_p$$

$$y_p = \rho_p \cdot \operatorname{sen}\theta_p$$

$$\operatorname{tg}\theta_p = \frac{y_p}{x_p} \qquad \boxed{x_p \neq 0}$$

$$\begin{cases} x_1' = p x_1 - q y_1 \\ y_1' = q x_1 + p y_1 \end{cases}$$

$$\boxed{x_{1(t)}^2 + y_{1(t)}^2 = \rho_{(t)}^2} \qquad \boxed{\operatorname{tg}\theta = \frac{y_{1(t)}}{x_{1(t)}}}$$

ponendo alle derivate ottengo

$$\cancel{2} x_1 x_1' + \cancel{2} y_1 y_1' = \cancel{2} \rho \rho'$$

$$x_1 \cdot (p x_1 - q y_1) + y_1 \cdot (q x_1 + p y_1) = \rho \cdot \rho'$$

$$x_1^2 p - \cancel{x_1 y_1 q} + \cancel{x_1 y_1 q} + p y_1^2 = \rho\rho'$$

$$p\underbrace{(x_1^2+y_1^2)}_{\rho^2} = \rho\cdot\rho'$$

$$\boxed{p\rho = \rho'} \qquad \boxed{\rho(t) = Ae^{pt}}$$

$$\mathrm{tg}(\theta(t)) = \frac{y_1(t)}{x(t)} \xrightarrow{\text{ponendo alle derivate}} \left(1+\mathrm{tg}^2\theta(t)\right)\cdot\theta'(t) =$$

$$= \frac{y_1'(t)\cdot x_1(t) - x_1'(t)\cdot y_1(t)}{x_1^2(t)}$$

Pertanto essendo $\begin{cases} x_1' = p x_1 - q y_1 \\ y_1' = q x_1 + p y_1 \end{cases}$

si ha con

$$\left(1 + tg^2\,\theta(t)\right) \cdot \theta'(t) = \frac{\overset{y_1'}{\overset{\shortparallel}{(q x_1 + p y_1)}}\, x_1 - \overset{x_1'}{\overset{\shortparallel}{(p x_1 - q y_1)}} \cdot y_1}{x_1^2(t)} =$$

$$tg(\theta) = \frac{y_1(t)}{x_1(t)} \qquad = \frac{q x_1^2 + p \cancel{x_1} y_1 - p \cancel{x_1} y_1 + q y_1^2}{x_1^2}$$

$$\boxed{\theta'(t) = q} \qquad = q\left(1 + \frac{y_1^2}{x_1^2}\right) = q \cdot \left(1 + tg^2\,\theta(t)\right)$$

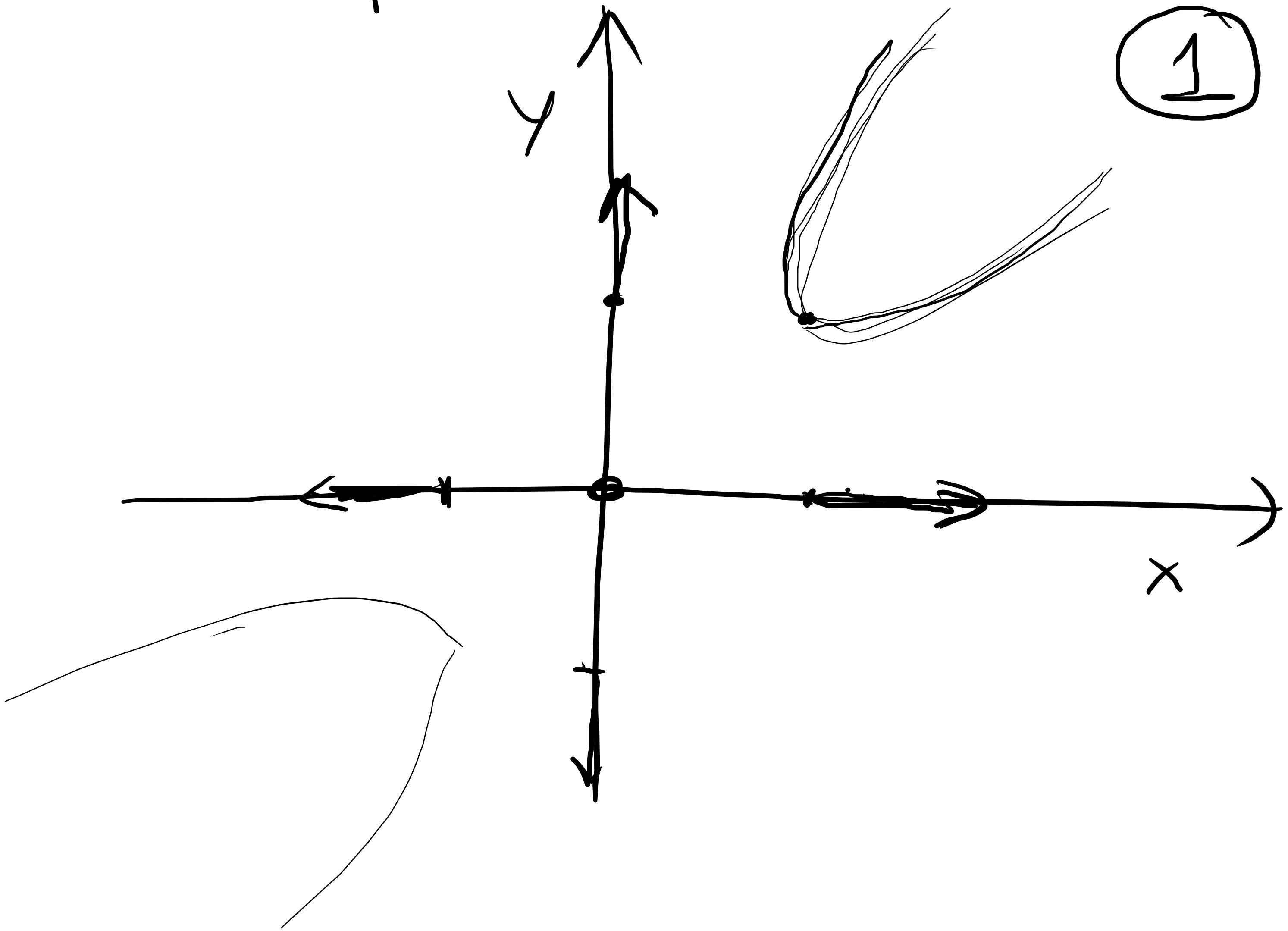
$$\theta(t) = (qt + q_0)$$

$$x_1(t) = \rho(t) \cdot \cos\theta(t) = \underline{A_1\, e^{\alpha t} \cos(qt + q_1)}$$

$$\rho(t) = A \cdot e^{\alpha t}$$

$$y_1(t) = \rho(t) \cdot \operatorname{sen}\theta(t) = \underline{A_2 \cdot e^{\alpha t} \operatorname{sen}(qt + q_2)}$$

Nel piano delle fasi come in $x(t), y(t)$



(1) $\Delta > 0$

m_1, m_2 reali

m_1, m_2 entrambi positivi

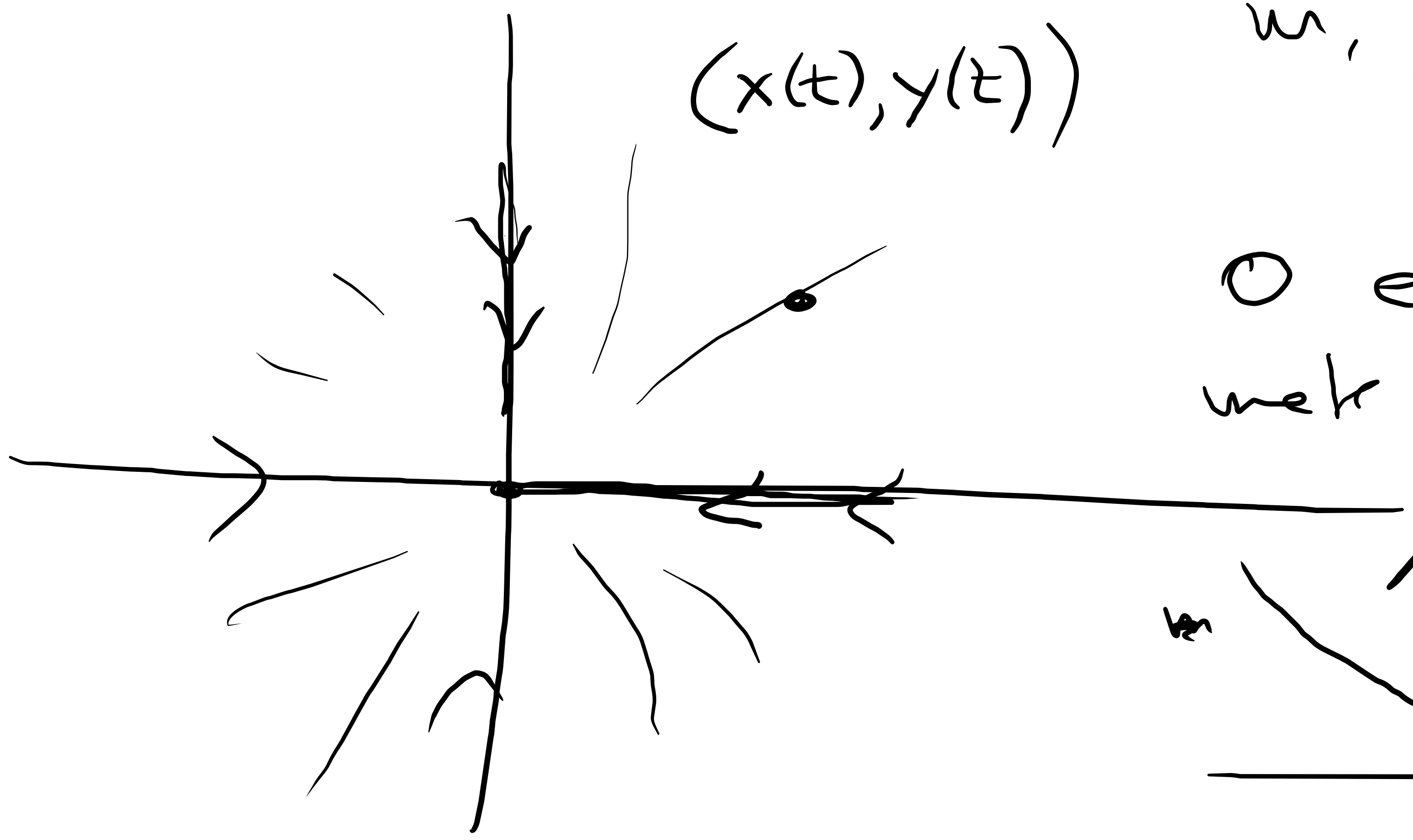
L'origine è instabile

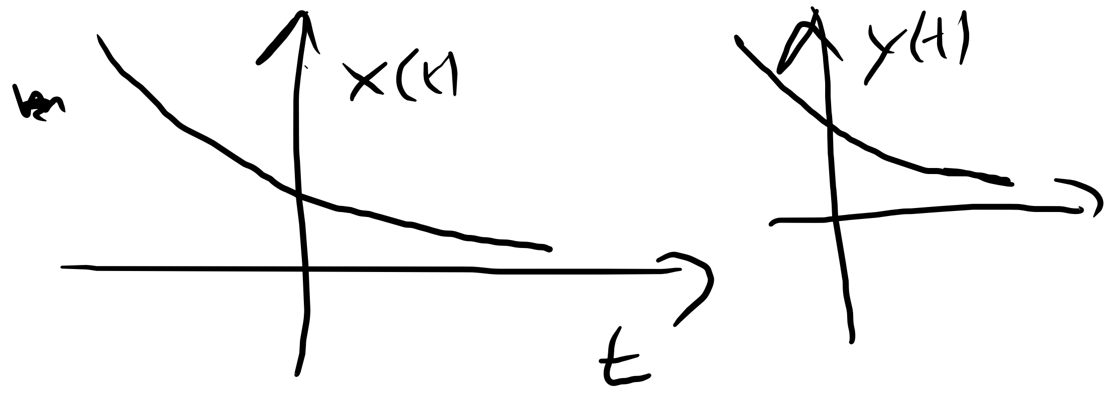
1

$\Delta > 0$

m_1, m_2 entrambi negativi

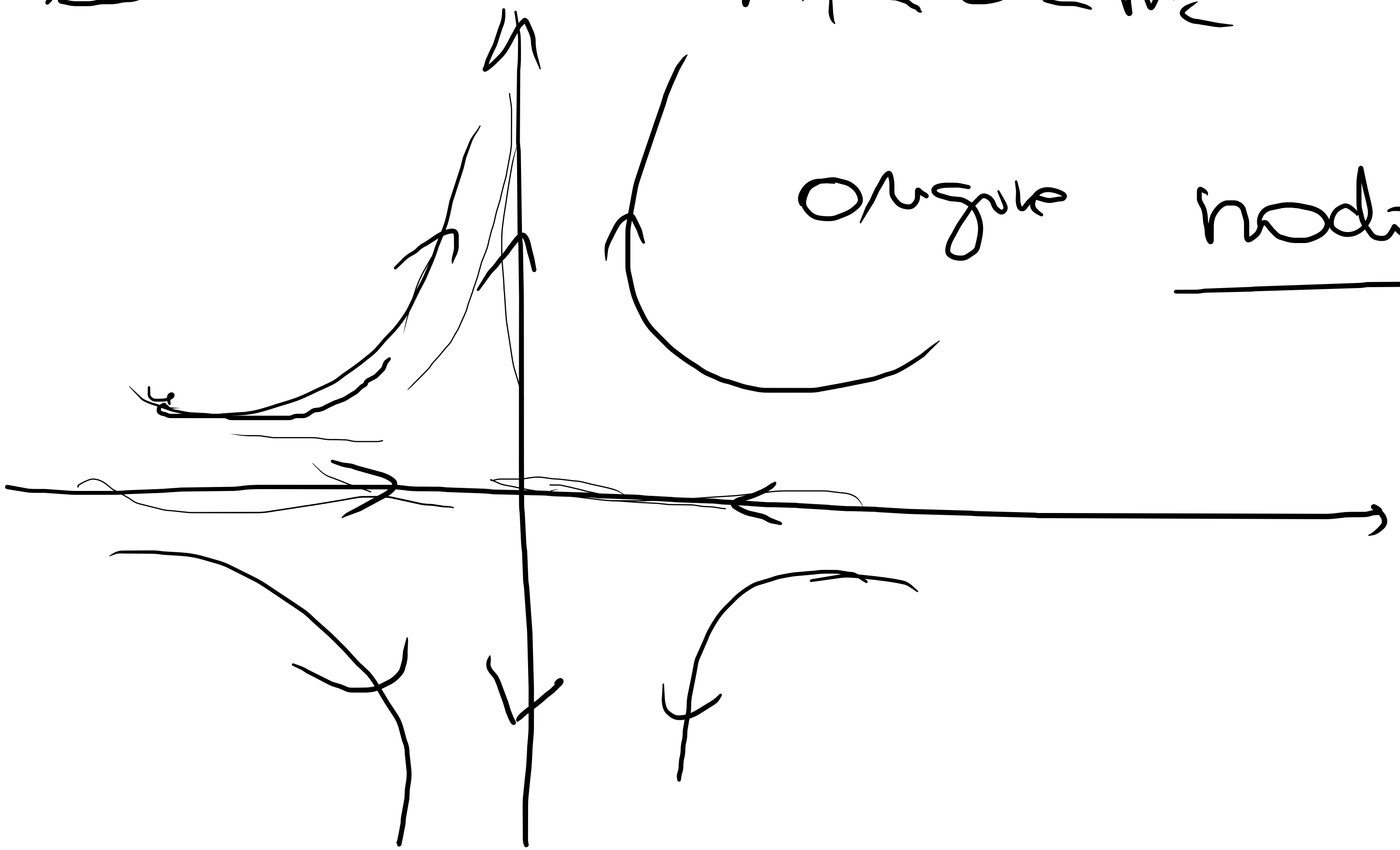
O è stabile e asintoticamente = nodo stabile





(1) $\Delta > 0$

$m_1 < 0 < m_2$



Origine **nodo INSTABILE**

2

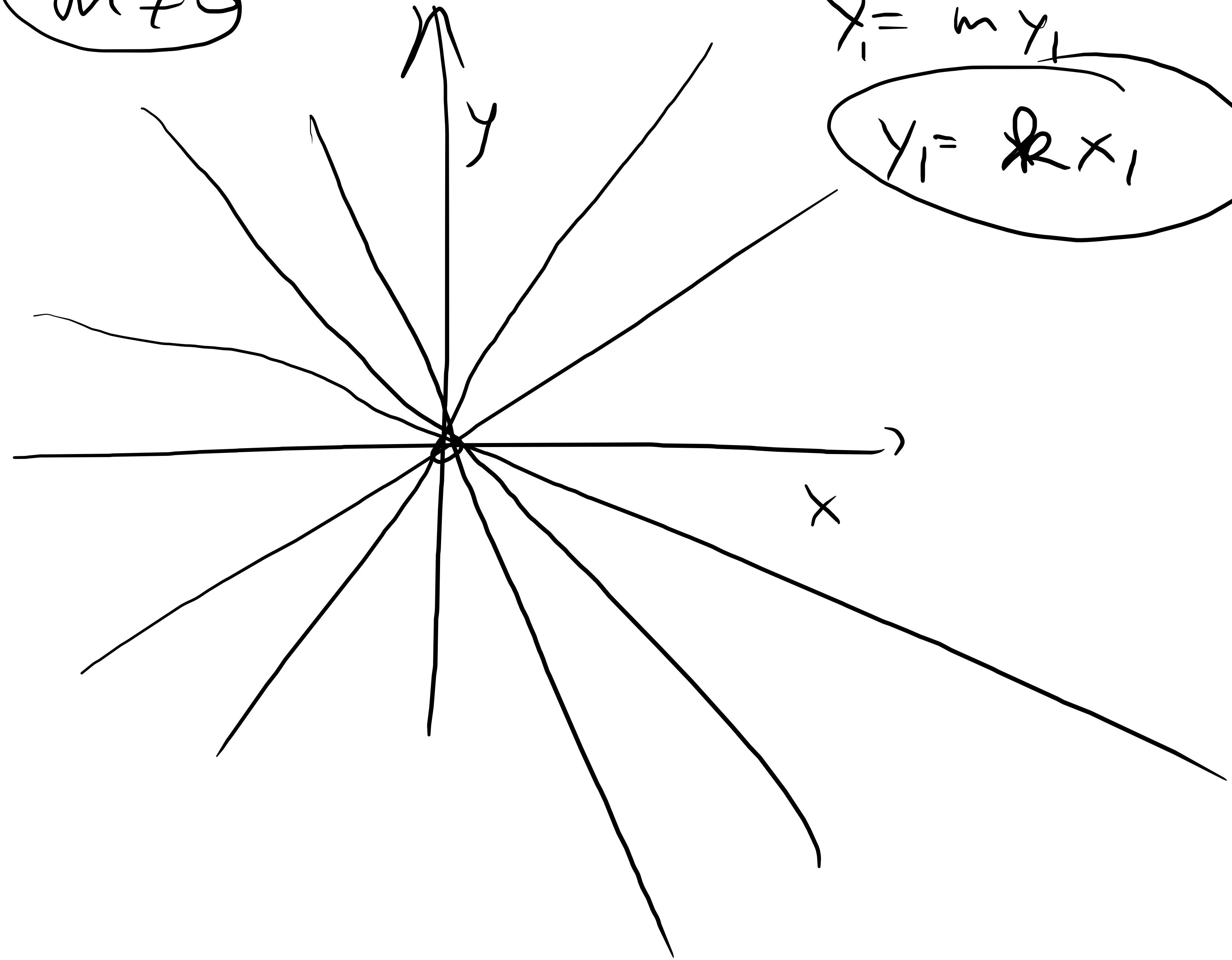
$\Delta = 0$

$m \neq 0$

2a) $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

$x_1' = m x_1$

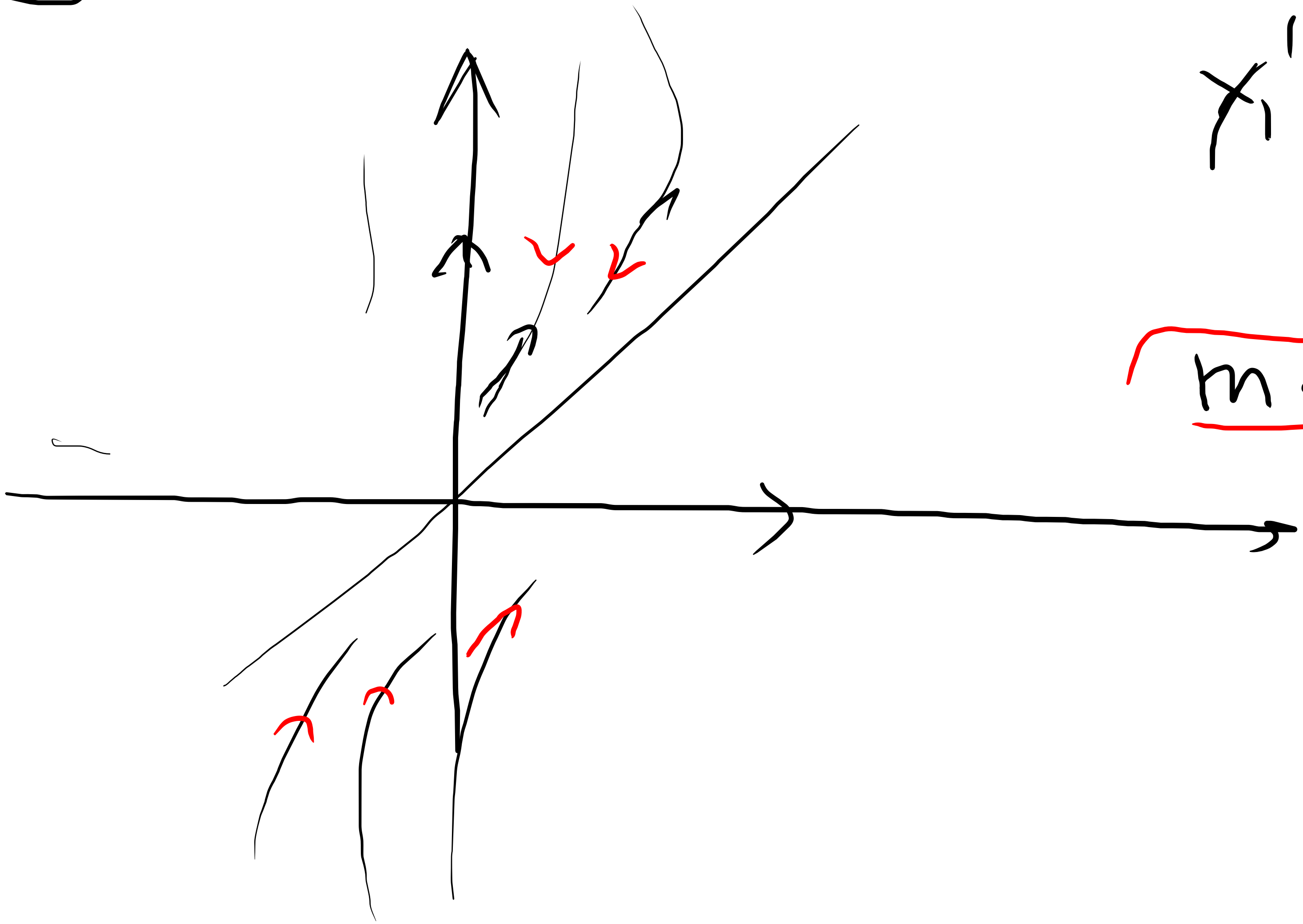
$y_1' = m y_1$



$y_1 = k x_1$

② $\Delta = 0$

2b)



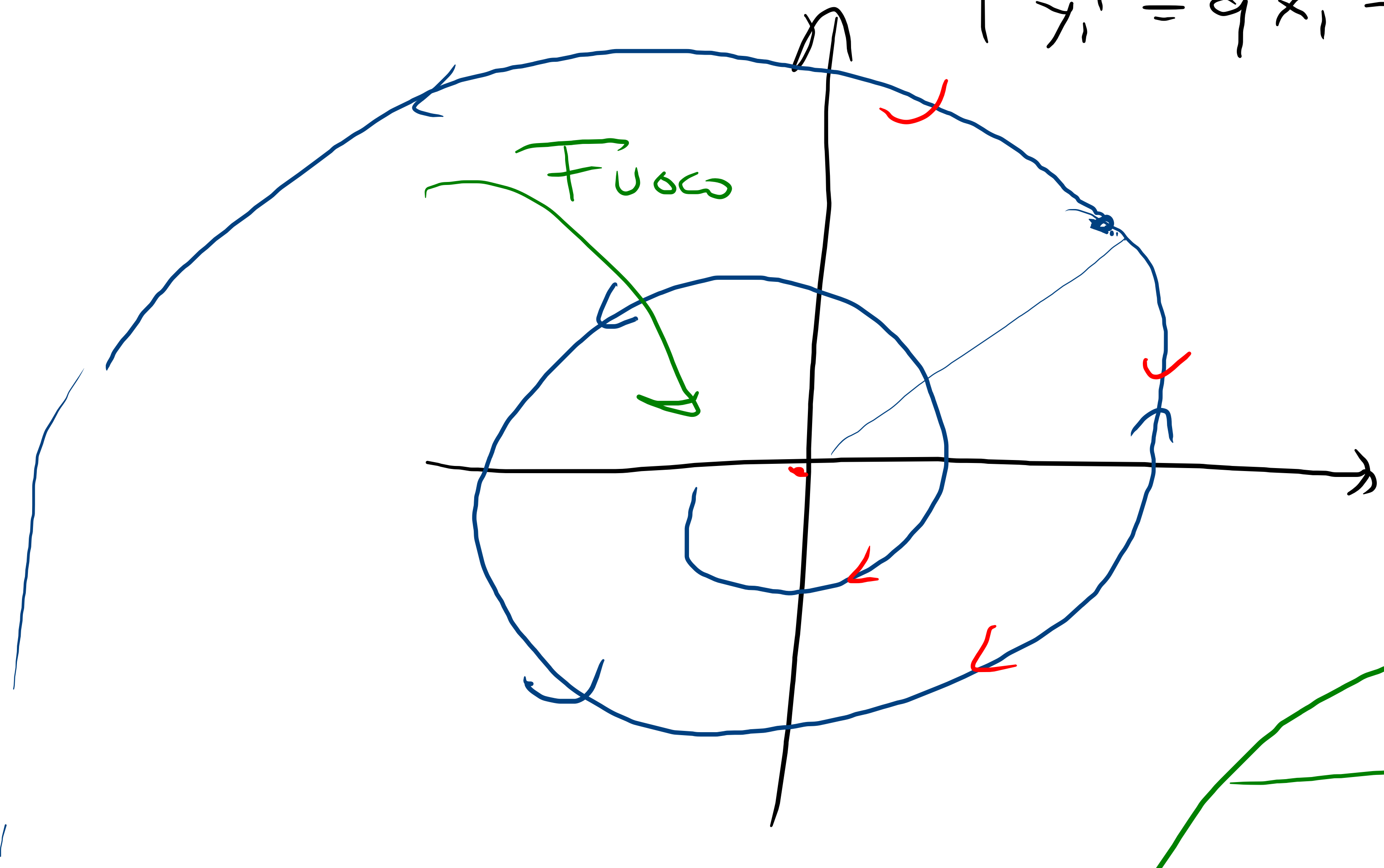
$$x_1' = m x_1 + y_1$$
$$y_1' = m y_1$$

$m > 0$

$m < 0$

③ $\Delta < 0$

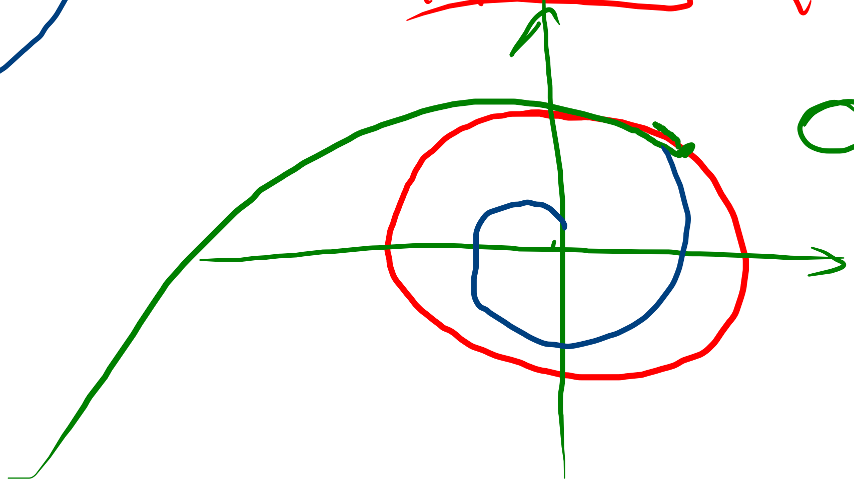
$$\begin{cases} x_1' = p x_1 + q y_1 \\ y_1' = q x_1 + p y_1 \end{cases} \rightsquigarrow \begin{cases} \rho' = p\rho \to \rho(t) = e^{pt} \\ \theta' = q \quad \theta(t) = qt + \theta_0 \end{cases}$$



$p > 0$ ORIGINE INSTABILE

$p < 0$ STABILE

$\boxed{p = 0}$ $\boxed{\rho' = 0}$



Origine CENTRO

Consideriamo due popolazioni interagenti la cui evoluzione demografica è descritta da un modello costituito da 2 eq. diff. lineari a coeff. costanti

$$\begin{cases} x'(t) = \underline{a\,x(t)} + b\,y(t) \\ y'(t) = \underline{c\,x(t)} + \underline{d\,y(t)} \end{cases}$$

se $a>0$ $\quad d>0 \quad \underline{b<0} \quad \underline{c<0}$ popolazioni in competizione

$\Delta = (a-d)^2 + \underline{4bc} > 0$ quindi $x(t) = A e^{m_1 t} + B e^{m_2 t}$

$y(t) = A_2 e^{m_1 t} + B_2' e^{m_2 t}$

popolazione in collaborazione

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

$b > 0 \quad a < 0$

$c > 0 \quad d < 0$

Due popolazioni modi con una è preda dell'altra

preda $\longrightarrow$

predatore $\longrightarrow$

$$\begin{cases} x' = ax + by \\ y' = \underline{c}x + dy \end{cases}$$

$a > 0 \quad \underline{b < 0}$

$\underline{c > 0} \quad d < 0$

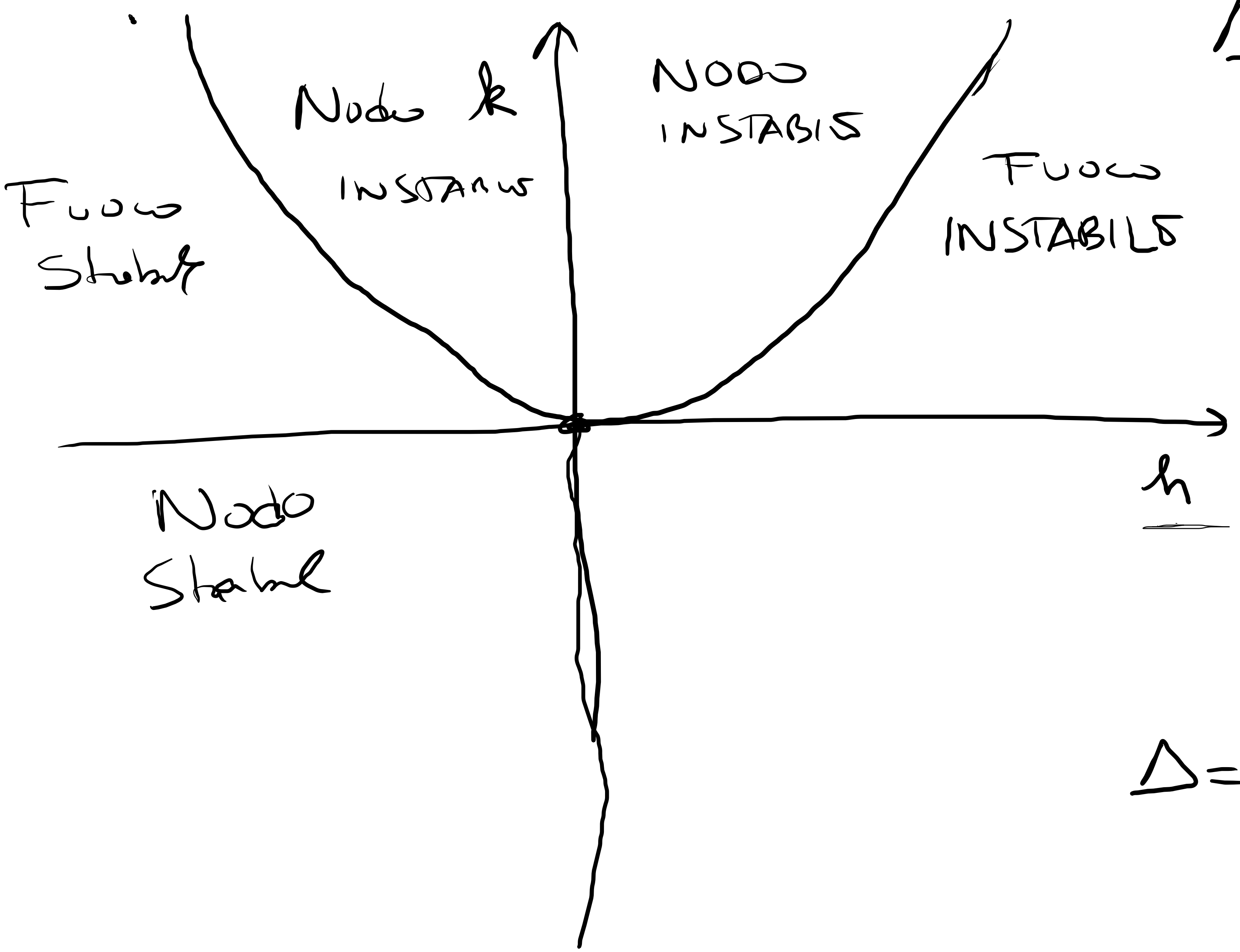
$$\Delta = (a-d)^2 + 4\underline{bc}$$

> 0

$= 0$

< 0

$h^2 - 4k$



$\Delta = 0 \qquad k = \dfrac{h^2}{4}$

$h = -(a+d)$

$k = ad - bc$

$m_1 = \dfrac{h}{2} \mp \dfrac{\sqrt{\Delta}}{2}$

$\Delta > 0$

$\Delta = h^2 - 4k > h^2$

se $k < 0$

Modelli lineari sono molto approssimativi

$$\begin{cases} x' = ax - bxy \\ y' = cx - dxy \end{cases}$$

è non lineare

ma descrive meglio il modello predatore-preda

$$\begin{cases} x' = \phi_1(x, y) \\ y' = \phi_2(x, y) \end{cases}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Se

ϕ_1 e ϕ_2 sono DIFFERENZIABILI in $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ punto di equilibrio del sistema

$$
\begin{cases}
x' = \cancel{\overset{\overset{0}{\|}}{\Phi_1(x_0, y_0)}} + \underbrace{\overset{\overset{a}{\|}}{\Phi_{1x}(x_0, y_0)}(x - x_0) + \overset{\overset{b}{\|}}{\Phi_{1y}(x_0, y_0)}(y - y_0)} + R_1 \\
y' = \cancel{\underset{\underset{0}{\|}}{\Phi_2(x_0, y_0)}} + \underset{\underset{c}{\|}}{\Phi_{2x}(x_0, y_0)}(x - x_0) + \Phi_{2y}(x_0, y_0)(y - y_0) + R_2
\end{cases}
$$