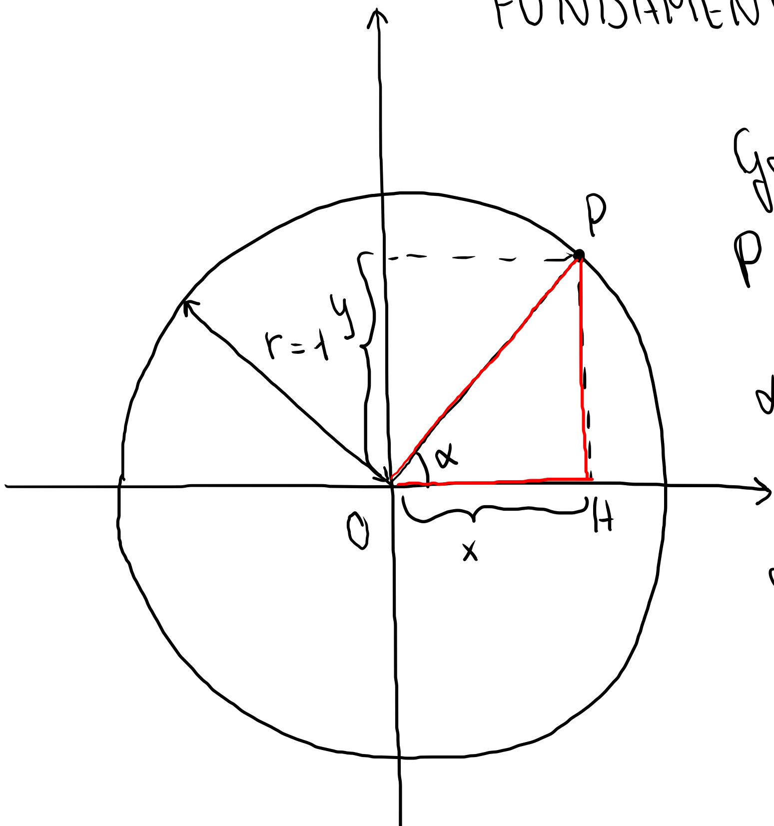
FUNDAMENTALS OF TRIGONOMETRY



Goniometric circumference : $r = 1$

$P(x, y) = (\cos\alpha, \sin\alpha)$

α : angle between the positive x - semiaxis and the segment OP

$\cos\alpha$: cosine of α

$\sin\alpha$: sin of α

$\overset{\triangle}{OHP}$ right triangle (right angle $\widehat{OHP}$)

By the Pythagorean Theorem, $x^2 + y^2 = 1$

$x = \cos\alpha$, $y = \sin\alpha$

$(\cos\alpha)^2 + (\sin\alpha)^2 = 1$

$$(\cos\alpha)^2 + (\sin\alpha)^2 = 1$$

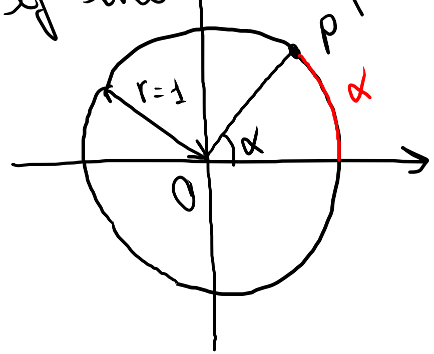
$$\boxed{\cos^2\alpha + \sin^2\alpha = 1 \quad \forall \alpha \in \mathbb{R}}$$

Fundamental identity in trigonometry

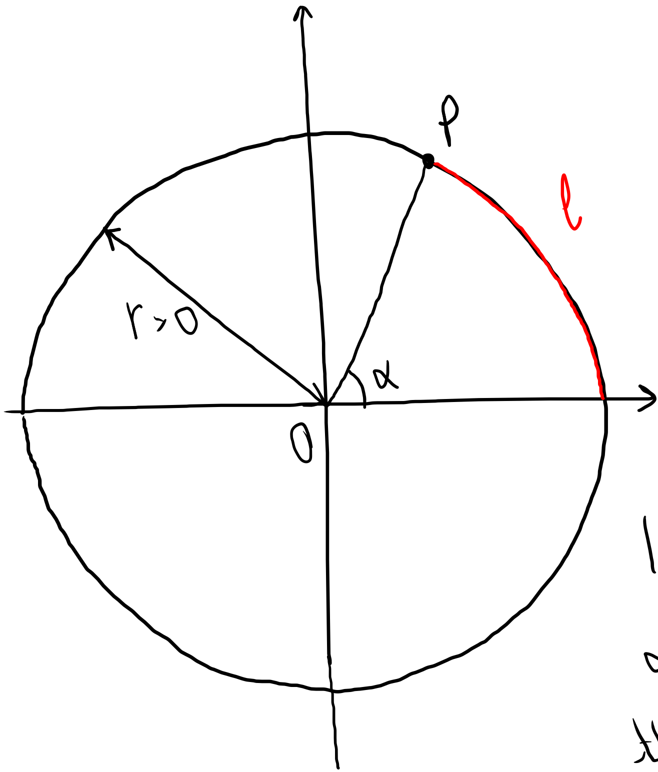
MEASURES OF ANGLES

An angle in trigonometry is measured in RADIANS.

In a goniometric circumference, an angle measures α radians if the corresponding arc has length α.



(goniometric circumference)



$$\alpha = \frac{\ell}{r}$$

(any circumference : $r > 0$)

In any circumference, an angle measures α radians if α is the ratio between the length of the corresponding arc, ℓ, and the length of the radius, r.

Angles in degrees	Angles in radians
360°	2π
180°	π
90°	$\frac{\pi}{2}$
$120^\circ = \frac{360^\circ}{3}$	$\frac{2\pi}{3} = \frac{2}{3}\pi$
60°	$\frac{\pi}{3}$
30°	$\frac{\pi}{3 \cdot 2} = \frac{\pi}{6}$
$45^\circ = \frac{90^\circ}{2}$	$\frac{\pi}{2 \cdot 2} = \frac{\pi}{4}$
β	$\frac{\beta}{360^\circ} \cdot 2\pi$

Definition of radian

1 radian is the measure of an angle which is associated to an arc whose length is $\frac{1}{2\pi}$ along the goniometric circumference.

Definition of tangent of α

$$tg\,\alpha = \frac{\sin\alpha}{\cos\alpha} \quad (\cos\alpha \neq 0)$$

Sine, cosine and tangent of α are not only numbers, but also functions.

1) DOMAINS

$$\boxed{\mathrm{dom}(\sin) = \mathbb{R} = \mathrm{dom}(\cos\alpha)}$$

because $\sin\alpha$ and $\cos\alpha$ are well-defined for any $\alpha \in \mathbb{R}$.

$$\mathrm{dom}(tg) = \{\alpha \in \mathbb{R} : \cos\alpha \neq 0\}$$

$$\cos\alpha = 0 \longleftrightarrow \alpha = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi, \frac{9}{2}\pi, \ldots$$

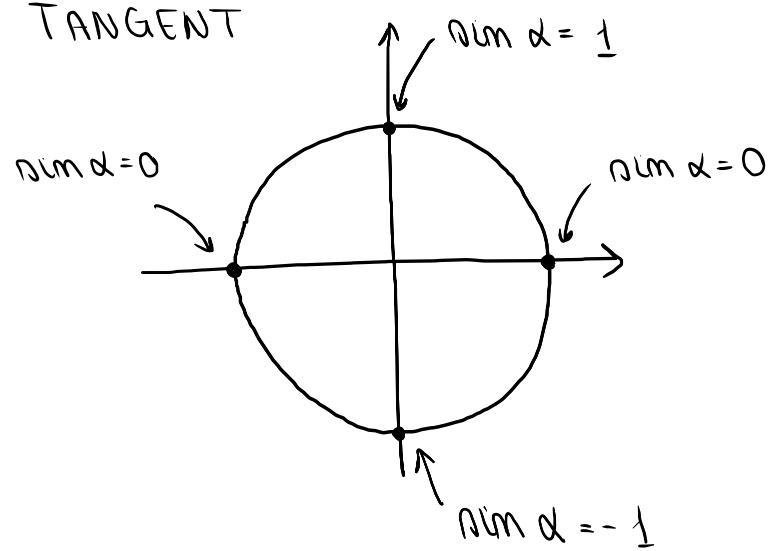
$$-\frac{\pi}{2}, -\frac{3}{2}\pi, -\frac{5}{2}\pi, -\frac{7}{2}\pi, -\frac{9}{2}\pi, \ldots$$

$$\longleftrightarrow \alpha = \frac{\pi}{2} + K\pi \qquad K \in \mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$$

$$\mathrm{dom}(tg) = \{\alpha \in \mathbb{R} : \alpha \neq \frac{\pi}{2} + K\pi, K \in \mathbb{Z}\}$$

$$= \mathbb{R} \setminus \{\frac{\pi}{2} + K\pi : K \in \mathbb{Z}\}$$

2) RELEVANT VALUES FOR SINE, COSINE, TANGENT



SINE

$\sin \alpha = 0$

$\updownarrow$

$\alpha = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots$

$-\pi, -2\pi, -3\pi, -4\pi, -5\pi, \ldots$

$\updownarrow$

$\alpha = k\pi \qquad k \in \mathbb{Z}$

$\sin \alpha = 1 \longleftrightarrow \alpha = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \frac{\pi}{2} + 6\pi, \ldots$

$\frac{\pi}{2} - 2\pi, \frac{\pi}{2} - 4\pi, \frac{\pi}{2} - 6\pi, \ldots$

$\longleftrightarrow \alpha = \frac{\pi}{2} + 2k\pi \qquad k \in \mathbb{Z}$

$\sin \alpha = -1 \Longleftrightarrow \alpha = \frac{3}{2}\pi, \frac{3}{2}\pi + 2\pi, \frac{3}{2}\pi + 4\pi, \frac{3}{2}\pi + 6\pi, \dots$

$\frac{3}{2}\pi - 2\pi, \frac{3}{2}\pi - 4\pi, \frac{3}{2}\pi - 6\pi, \dots$

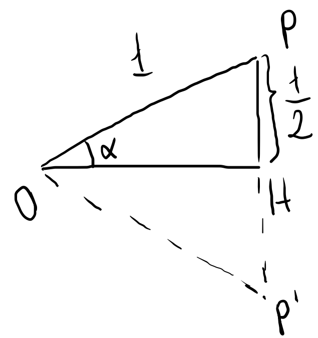
$\Longleftrightarrow \alpha = \frac{3}{2}\pi + 2k\pi \qquad k \in \mathbb{Z}$

$\sin\alpha = -1 \longleftrightarrow \alpha = \frac{3}{2}\pi, \frac{3}{2}\pi + 2\pi, \frac{3}{2}\pi + 4\pi, \frac{3}{2}\pi + 6\pi, \dots$

$\frac{3}{2}\pi - 2\pi, \frac{3}{2}\pi - 4\pi, \frac{3}{2}\pi - 6\pi, \dots$

$$\longleftrightarrow \alpha = \frac{3}{2}\pi + 2K\pi \qquad K \in \mathbb{Z}$$

$\sin\alpha = \frac{1}{2} \longleftrightarrow \alpha = \frac{\pi}{6} + 2K\pi$ or $\alpha = \frac{5}{6}\pi + 2K\pi$ $\quad K \in \mathbb{Z}$



$\overset{\triangle}{OHP}$ right triangle is half an equilateral triangle

$\longrightarrow \alpha = \frac{60^\circ}{2} = 30^\circ = \frac{\pi}{6}$

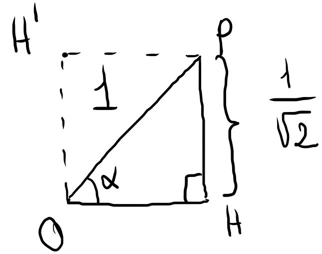
$\sin\alpha = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\Updownarrow$

$\alpha = \frac{\pi}{4} + 2K\pi$ or

$\alpha = \frac{3}{4}\pi + 2K\pi$

$K \in \mathbb{Z}$



$\overset{\Delta}{OHP}$ right triangle is half a square

$\rightarrow \alpha = \frac{90°}{2} = 45° = \frac{\pi}{4}$

$\forall \alpha \in \mathbb{R}, \quad -1 \leq \sin\alpha \leq 1 \longleftrightarrow |\sin\alpha| \leq 1$

$\boxed{\text{COSINE}}$

$\cos\alpha = 0 \longleftrightarrow \alpha = \frac{\pi}{2} + K\pi \quad K \in \mathbb{Z}$

$\cos\alpha = -1 \longleftrightarrow \alpha = \pi + 2K\pi \quad K \in \mathbb{Z}$

$\cos\alpha = 1 \longleftrightarrow \alpha = 2K\pi \quad K \in \mathbb{Z}$

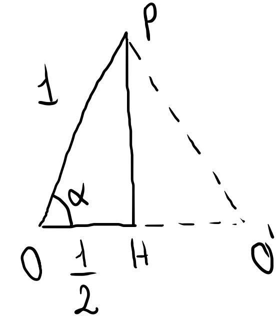
$\cos \alpha = \frac{1}{2}$

$\Updownarrow$

$\alpha = \frac{\pi}{3} + 2K\pi$ or

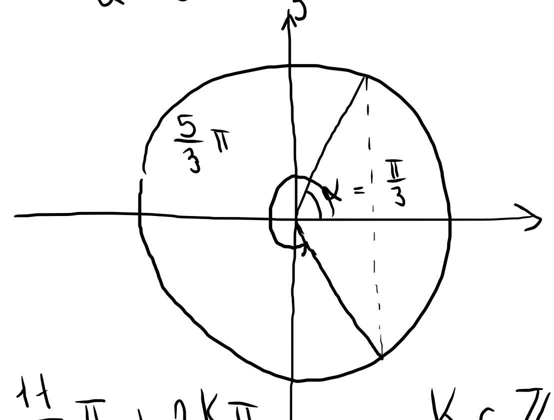
$\alpha = \frac{5}{3}\pi + 2K\pi$

$K \in \mathbb{Z}$



$\overset{\triangle}{OHP}$ half an equilateral triangle $\overset{\triangle}{OO'P}$

$\rightarrow \alpha = 60° = \frac{\pi}{3}$



$K \in \mathbb{Z}$

$\cos \alpha = \frac{\sqrt{3}}{2} \longleftrightarrow \alpha = \frac{\pi}{6} + 2K\pi$ or $\alpha = \frac{11}{6}\pi + 2K\pi$

$\forall \alpha \in \mathbb{R}, \quad -1 \leq \cos\alpha \leq 1 \longleftrightarrow |\cos\alpha| \leq 1$

$\boxed{\text{TANGENT}}$

$tg\,\alpha = 0 \longleftrightarrow \dfrac{\sin\alpha}{\cos\alpha} = 0 \longleftrightarrow \sin\alpha = 0 \longleftrightarrow \alpha = K\pi$

$K \in \mathbb{Z}$

$tg\,\alpha = 1 \longleftrightarrow \sin\alpha = \cos\alpha \longleftrightarrow \alpha = \dfrac{\pi}{4} + K\pi \quad K \in \mathbb{Z}$

$tg\,\alpha = -1 \longleftrightarrow \sin\alpha = -\cos\alpha \longleftrightarrow \alpha = \dfrac{3}{4}\pi + K\pi \quad K \in \mathbb{Z}$

$tg\,\alpha = \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \longleftrightarrow \alpha = \dfrac{\pi}{6} + K\pi \quad K \in \mathbb{Z}$

$tg\,\alpha = \sqrt{3} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \longleftrightarrow \alpha = \dfrac{\pi}{3} + K\pi \quad K \in \mathbb{Z}$

$$\left.\begin{array}{l} tg\big|_{\left]-\frac{\pi}{2},\frac{\pi}{2}\right[} \text{ is increasing} \\[2ex] \lim\limits_{\alpha\to\left(-\frac{\pi}{2}\right)^+} tg\,\alpha = -\infty \\[2ex] \lim\limits_{\alpha\to\left(\frac{\pi}{2}\right)^-} tg\,\alpha = +\infty \\[2ex] tg\big|_{\left]-\frac{\pi}{2},\frac{\pi}{2}\right[} \text{ is continuous} \end{array}\right\} \xrightarrow[\text{value theorem}]{\text{Intermediate}} \begin{array}{c} \operatorname{Im}\left(tg\big|_{\left]-\frac{\pi}{2},\frac{\pi}{2}\right[}\right) \\ \| \\ \mathbb{R} \end{array}$$

(the restriction of tg to $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ attains all the real values)

PROPERTIES OF SINE, COSINE, TANGENT FUNCTIONS

$\boxed{\text{SINE}}$

$\sin : \mathbb{R} \to [-1,1]$ is continuous (1)

periodic with minimum period 2π: (2)

$\forall x \in \mathbb{R}, \ \sin(x + 2\pi) = \sin x$

$\longrightarrow \forall x \in \mathbb{R}, \ \sin(x + 2k\pi) = \sin x$

odd function: (3)

$\forall x \in \mathbb{R}, \ \sin(-x) = -\sin x$

$\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ is increasing (so it is strictly monotone) (4)

$\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1,1]$ is injective and surjective

$\longrightarrow \sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ is bijective and then it is invertible (5) arcsine is the inverse of the restriction of the sine to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$