Properties of the trigonometric functions

$\cos : \mathbb{R} \to [-1,1]$ is continuous (1)

periodic with minimum period 2π (2)

$\cos(x+2\pi) = \cos x \quad \forall x \in \mathbb{R}$

$\longrightarrow \cos(x+2k\pi) = \cos x \quad \forall x \in \mathbb{R},\ \forall k \in \mathbb{Z}$

(3)

is even

$\cos(-x) = \cos x, \quad \forall x \in \mathbb{R}$

decreasing in $[0,\pi]$, then it is (4)

strictly monotone in $[0,\pi]$, then it is

injective in $[0,\pi]$

$\cos|_{[0,\pi]}$ attains all the values in $[-1,1]$ (5)

$\cos|_{[0,\pi]} : [0,\pi] \longrightarrow [-1,1]$ is injective and surjective

that is, it is bijective, then it is invertible.

$\arccos : [-1,1] \longrightarrow [0,\pi]$ is the inverse function of $\cos|_{[0,\pi]}$

TANGENT

$tg : \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\} \longrightarrow \mathbb{R}$

is continuous (1)

is periodic with minimum period π (2)

$tg(x+\pi) = tg\, x, \ \forall x \in dom(tg)$

$\longrightarrow tg(x+k\pi) = tg\, x, \ \forall x \in dom(tg)$
$\forall k \in \mathbb{Z}$

is odd $\quad tg(-x) = -tg\, x, \ \forall x \in dom(tg)$ (3)

$tg|_{]-\frac{\pi}{2},\frac{\pi}{2}[}$ is increasing, then injective

$tg|_{]-\frac{\pi}{2},\frac{\pi}{2}[}$ attains all the values in $\mathbb{R}$

Then $tg|_{]-\frac{\pi}{2},\frac{\pi}{2}[} :]-\frac{\pi}{2},\frac{\pi}{2}[\longrightarrow \mathbb{R}$ is bijective, then invertible.

$arctg : \mathbb{R} \longrightarrow]-\frac{\pi}{2},\frac{\pi}{2}[$ is the inverse function of $tg|_{]-\frac{\pi}{2},\frac{\pi}{2}[}$

LIMITS

$f: X \longrightarrow Y \qquad X, Y \subset \mathbb{R}$

x_0 accumulation point of X

($\forall\ U_{x_0}$ neighbourhood of x_0, $\exists\ x \in X \setminus \{x_0\}$ such that $x \in U_{x_0}$)

$\lim_{x \to x_0} f(x) = \ell$ <u>Definition of limit of the function f</u>

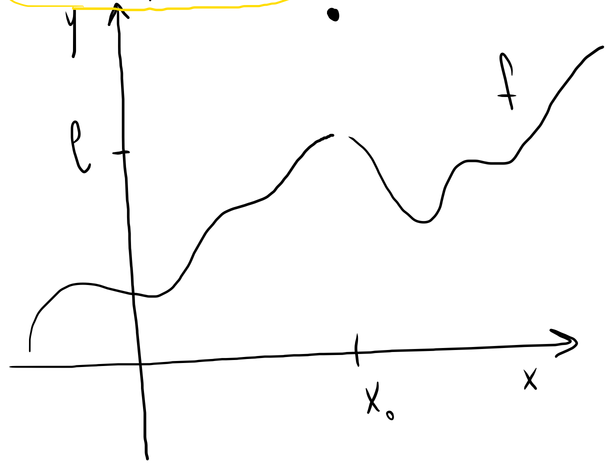
if $\forall\ V_\ell$ neighbourhood of ℓ,

$\exists\ U_{x_0}$ neighbourhood of x_0,

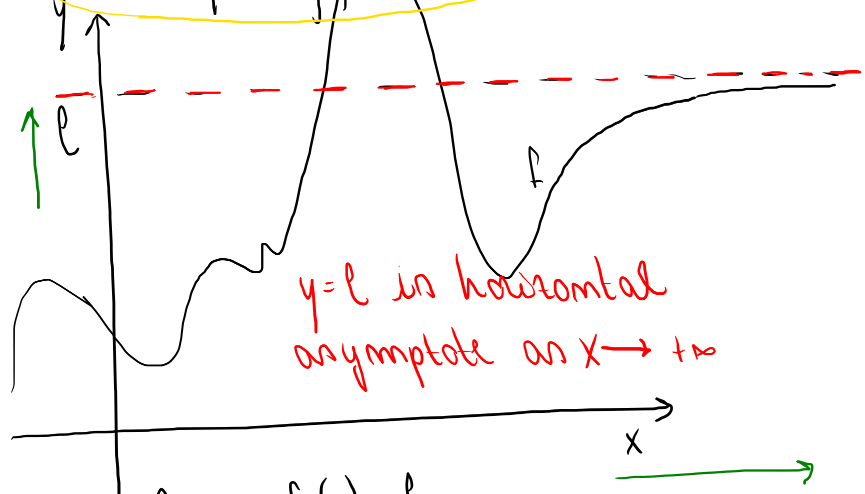
such that, $\forall\ x \in X \setminus \{x_0\} \cap U_{x_0}$, $f(x) \in V_\ell$

$x_0, \ell \in \mathbb{R} \cup \{\pm\infty\}$

$x_0 \in \mathbb{R},\ \ell \in \mathbb{R}$

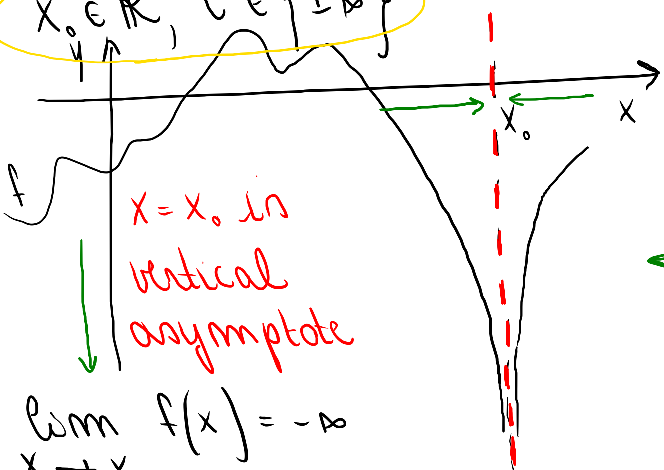


$x_0 \in \{\pm\infty\},\ \ell \in \mathbb{R}$



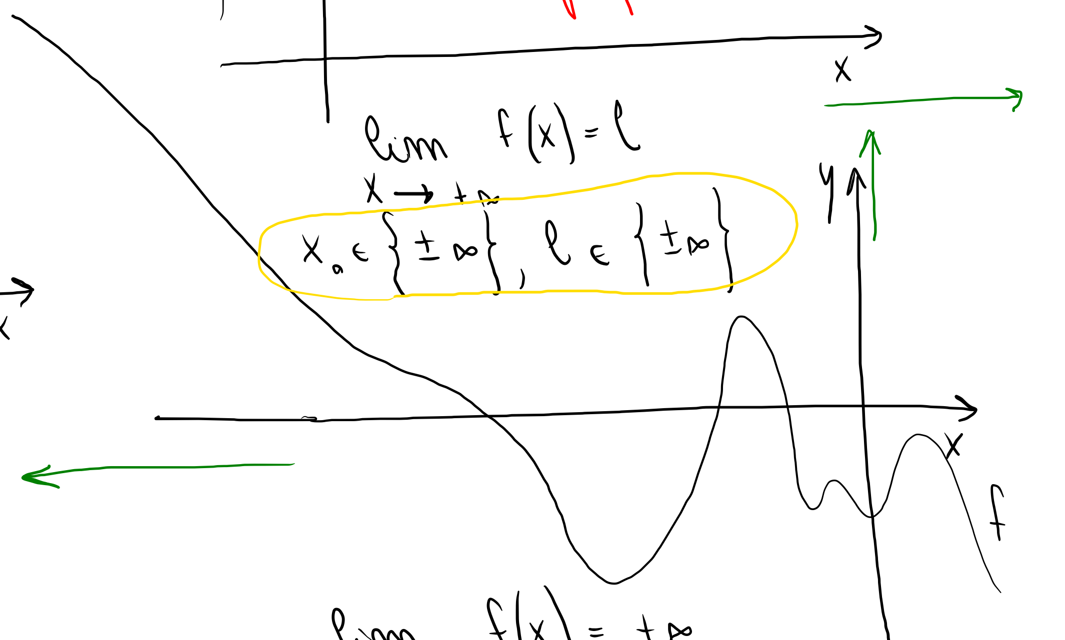
$\lim_{x \to +\infty} f(x) = \ell$

$x_0 \in \mathbb{R},\ \ell \in \{\pm\infty\}$



$\lim_{x \to x_0} f(x) = -\infty$

$x_0 \in \{\pm\infty\},\ \ell \in \{\pm\infty\}$



$\lim_{x \to -\infty} f(x) = +\infty$

Property

$f : X \to Y$ $\quad x_0 \in X$ $\quad X, Y \subset \mathbb{R}$

x_0 accumulation point of X

Then f is continuous at x_0 $\longleftrightarrow$ $\lim_{x \to x_0} f(x) = f(x_0)$

EXERCISES

1) $\lim_{x \to 2} \sqrt{3x+3}$

$\parallel$

$\sqrt{3 \cdot 2 + 3} = \sqrt{9} = 3$

$f(x) = \sqrt{3x+3}$

$dom(f) = \{x \in \mathbb{R} : 3x + 3 \geq 0\}$

$= [-1, +\infty[$

f is continuous in $dom(f)$

2) $\lim_{x \to -2} \frac{x^2-4}{x+2} = \lim_{x \to -2} (x-2)$

$x^2-4 = (x+2)(x-2)$

$[a^2-b^2 = (a+b)(a-b) \quad \forall a,b \in \mathbb{R}]$

$f(x) = \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}} = x-2$

$\text{dom}(f) = \mathbb{R}$

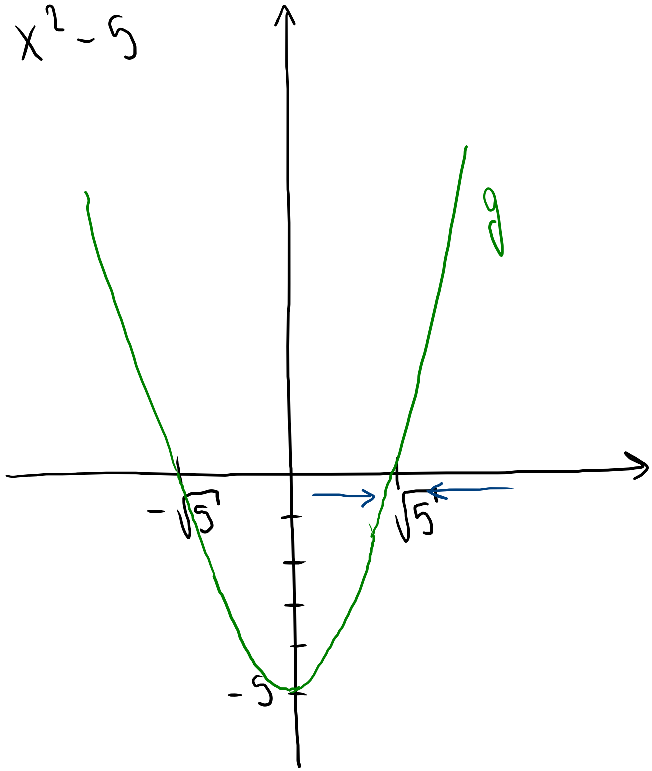
f is continuous in $\mathbb{R}$

$\lim_{x \to -2} (x-2) = -2-2 = -4$

3) $\lim_{x \to \sqrt{5}} \frac{1}{x^2-5}$

$f(x) = \frac{1}{x^2-5}$

$g(x) = x^2 - 5$



$$\lim_{x \to \sqrt{5}^-} (x^2 - 5) = 0^- \qquad \text{(a)}$$

$$\lim_{x \to \sqrt{5}^+} (x^2 - 5) = 0^+ \qquad \text{(b)}$$

(a) by the limit of reciprocal function, $\displaystyle\lim_{x \to \sqrt{5}^-} \frac{1}{x^2 - 5} = -\infty$

$\neq$

(b) $\longrightarrow \displaystyle\lim_{x \to \sqrt{5}^+} \frac{1}{x^2 - 5} = +\infty$

By the limit of the restriction,

$$\nexists \lim_{x \to \sqrt{5}} \frac{1}{x^2 - 5}$$

4) $\lim_{x \to 0} \sqrt[3]{1 + \frac{1}{x}}$

$f(x) = \sqrt[3]{1 + \frac{1}{x}}$

$\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$

$x \overset{f_1}{\longmapsto} \underbrace{1 + \frac{1}{x}}_{= t} \overset{f_2}{\longmapsto} \sqrt[3]{t}$

$f(x) = f_2 \circ f_1(x)$ compound function

Consider $\lim_{x \to 0^-} f(x)$ ⓐ

$\lim_{x \to 0^+} f(x)$ ⓑ

ⓐ $\lim_{x \to 0^-} f_1(x) = \lim_{x \to 0^-} \left(1 + \underbrace{\frac{1}{x}}_{\to -\infty}\right) = -\infty$

$\lim_{t \to -\infty} f_2(t) = \lim_{t \to -\infty} \sqrt[3]{t} = -\infty$

By the limit of the compound function,

$\lim_{x \to 0^-} f(x) = -\infty$

5) $\lim\limits_{x \to +\infty} \sin\left(\frac{x}{2}\right)$

$\operatorname{dom}(f) = \mathbb{R}$

$f(x) = \sin\left(\frac{x}{2}\right) = f_2 \circ f_1(x)$

compound function

$x \xmapsto{f_1} \frac{x}{2} \xmapsto{f_2} \sin(t)$

$\frac{x}{2} = t$

$\lim\limits_{x \to -\infty} f_1(x) = \lim\limits_{x \to -\infty} \frac{x}{2} = -\infty$

$\lim\limits_{t \to -\infty} f_2(t) = \lim\limits_{t \to -\infty} \sin(t)$

sin oscillates between -1 and 1

1)

Consider $\sin|_{\{t \in \mathbb{R} :\ t = 2k\pi,\ k \in \mathbb{Z}\}} = 0$ (constant function with value 0)

$\lim\limits_{t \to -\infty} \sin|_{\{t = 2k\pi,\ k \in \mathbb{Z}\}}(t) = \lim\limits_{t \to -\infty} 0 = 0$

2) Consider $\sin\Big|_{\left\{t=\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}\right\}} \equiv 1$

Then $\lim\limits_{t\to-\infty} \sin\Big|_{\left\{t=\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}\right\}}(t) = \lim\limits_{t\to-\infty} 1 = 1$

By the limit of the restriction, $\nexists \lim\limits_{x\to-\infty} \sin\left(\frac{x}{2}\right)$