MATHEMATICS CLASS

from December 3 to December 17, 2020

Exercise 1. Compute the derivatives of the following functions, specifying their domain and that of their derivative.

1) $f(x) = \dfrac{2+x}{3-x^2}$ 2) $f(x) = (5x^4 - \pi x - 1)^3$ 3) $f(x) = x^2|x|$

4) $f(x) = e^{\frac{2x}{x+1}}$ 5) $f(x) = \arctan(x^2)$ 6) $f(x) = \sqrt[3]{1+\sqrt{x^2-1}}$

7) $f(x) = \log\left(\dfrac{x^2-2}{x^2+x}\right)$ 8) $f(x) = \log\big(\log(\log x)\big)$ 9) $f(x) = \log(\arcsin(x-1))$

10) $f(x) = (1-2\cos x)^x$ 11) $f(x) = x^2 + \sqrt{2-\log(x-2)}$ 12) $f(x) = \sqrt[6]{\dfrac{3^{x+2}-1}{x^3-x}}$

13) $f(x) = \log(e^{-x}-1)$ 14) $f(x) = (\log_3 x - \log_9 |x+3|)^x$

Solutions

Notation. In the following we name A the domain of the function f, B the domain of its derivative f'.

1) $A = B =]-\infty, -\sqrt{3}[\cup]-\sqrt{3}, \sqrt{3}[\cup]\sqrt{3}, +\infty[, \quad f'(x) = \dfrac{x^2+4x+3}{(3-x^2)^2}$

2) $A = B = \mathbb{R}, \quad f'(x) = 3(5x^4 - \pi x - 1)^2(20x^3 - \pi)$

3) $A = B = \mathbb{R}, \quad f'(x) = \begin{cases} -3x^2 & \text{if } x < 0, \\ 3x^2 & \text{if } x \geq 0 \end{cases}$

4) $A = B =]-\infty, -1[\cup]-1, +\infty[, \quad f'(x) = \dfrac{2}{(x+1)^2} e^{\frac{2x}{x+1}}$

5) $A = B = \mathbb{R}, \quad f'(x) = \dfrac{2x}{1+x^4}$

6) $A =]-\infty, -1] \cup [1, +\infty[, \; B =]-\infty, -1[\cup]1, +\infty[, \quad f'(x) = \dfrac{x}{3\sqrt[3]{\left(1+\sqrt{x^2-1}\right)^2} \cdot \sqrt{x^2-1}}$

7) $A =]-\infty, -\sqrt{2}[\cup]-1, 0[\cup]\sqrt{2}, +\infty[, \quad f'(x) = \dfrac{x^2+4x+2}{(x^2-2)(x^2+x)}$

$B =]-\infty, -\sqrt{2}[\cup]-\sqrt{2}, -1[\cup]-1, 0[\cup]0, \sqrt{2}[\cup]\sqrt{2}, +\infty[$

8) $A =]e, +\infty[, \; B =]1, e[\cup]e, +\infty[\quad f'(x) = \dfrac{1}{x \log x \, \log(\log x)}$

9) $A =]1, 2], \; B =]0, 1[\cup]1, 2[\quad f'(x) = \dfrac{1}{\sqrt{1-(x-1)^2} \arcsin(x-1)}$

10) $A = B = \bigcup_{k \in \mathbb{Z}} \left] \dfrac{\pi}{3} + 2k\pi, \dfrac{5}{3}\pi + 2k\pi \right[$

$f'(x) = e^{x \log(1-2\cos x)} \left(\log(1-2\cos x) + \dfrac{2x \sin x}{1-2\cos x} \right)$

11) $A =]2, 2+e^2], \; B =]2, 2+e^2[\quad f'(x) = 2x - \dfrac{1}{2(x-2)\sqrt{2-\log(x-2)}}$

12) $A =]-\infty, -2] \cup]-1, 0[\cup]1, +\infty[$, $B =]-\infty, -2[\cup]-1, 0[\cup]1, +\infty[$

$$f'(x) = \frac{1}{6}\sqrt[6]{\left(\frac{x^3 - x}{3^{x+2} - 1}\right)^5} \cdot \frac{3^{x+2}(x^3 - x)\log 3 - (3x^2 - 1)(3^{x+2} - 1)}{(x^3 - x)^2}$$

13) $A =]-\infty, 0[$, $B = \mathbb{R} \setminus \{0\}$ $\quad f'(x) = \dfrac{1}{e^x - 1}$

14) $A = B = \left]\dfrac{1 + \sqrt{13}}{2}, +\infty\right[$, $\quad f'(x) = (\log_3 x - \log_9 |x + 3|)^x \cdot$

$$\cdot \left(\log(\log_3 x - \log_9 |x + 3|) + \frac{x}{\log_3 x - \log_9 |x + 3|} \cdot \left(\frac{1}{x \log 3} - \frac{1}{(x - 3)\log 9}\right)\right)$$

Exercise 2. Write the equation of the tangent to the graph of the function f at the point whose abscissa is given.

Function	xe^{x^2}	$\sin\left(3x^5 + \frac{\pi}{2}\right)$
Abscissa of the point	$x = 1$	$x = 0$

Solutions

- $y = e(3x - 2)$
- $y = 1$

Exercise 3. Determine the image set of each function and determine its inverse function.

$$f(x) = \begin{cases} \log(x + 1) & \text{if } -1 < x \leq 0 \\ e^{-x} & \text{if } x > 0 \end{cases}, \qquad g(x) = \begin{cases} 2x & \text{if } x < 0 \\ \log(x + 1) & \text{if } x \geq 0 \end{cases}$$

Solutions

$$f^{-1} :]-\infty, 1[\to]-1, +\infty[\qquad f^{-1}(y) = \begin{cases} e^y - 1 & \text{if } y \leq 0 \\ -\log y & \text{if } 0 < y < 1 \end{cases}$$

$$g^{-1} : \mathbb{R} \to \mathbb{R} \qquad g^{-1}(y) = \begin{cases} \frac{y}{2} & \text{if } y \geq 0 \\ e^y - 1 & \text{if } y < 0 \end{cases}$$

Exercise 4. Compute the following limits, using L'Hôpital's theorem, when it is the case.

1) $\displaystyle\lim_{x \to 0^-} \frac{1 + 2^{\frac{1}{x}}}{3 + 2^{\frac{1}{x}}}$, 2) $\displaystyle\lim_{x \to 0^+} \frac{1 + 2^{\frac{1}{x}}}{3 + 2^{\frac{1}{x}}}$, 3) $\displaystyle\lim_{x \to 1^-} \frac{\sqrt{1 - x^2}}{\arccos x}$,

4) $\displaystyle\lim_{x \to 0^+} \frac{x^x - 1}{x}$, 5) $\displaystyle\lim_{x \to -\infty} \frac{\log(1 + e^x)}{e^x}$, 6) $\displaystyle\lim_{x \to +\infty} \left(1 + \frac{a}{x}\right)^x$,

7) $\displaystyle\lim_{x \to 0} \frac{\arctan x - x}{\sin^3 x}$

Solutions

$$1)\ \frac{1}{3}, \qquad 2)\ 1, \qquad 3)\ 1,$$

$$4)\ -\infty, \qquad 5)\ 1, \qquad 6)\ e^a,$$

$$7)\ -\frac{1}{3}.$$

Exercise 5. For each of the following functions, determine the domain, possible symmetries, the sign, possible intersections with the axes, the limits at the extreme points of the domain. Finally, use the information obtained to draw the qualitative graphs.

$$1)\ f(x) = \sqrt{\frac{x^4-1}{x^2-4}}$$

$$2)\ g(x) = \begin{cases} \log(-x-1) & \text{if } x < -1, \\ \sqrt{x-1}-1 & \text{if } x \geq 1 \end{cases}$$

Solutions

1) The *domain* of f is $A =]-\infty,-2[\cup [-1,1] \cup]2,+\infty[$.

   Symmetries:

   The function is even.

   Sign and intersection with the axes:

   $$f(x) \geq 0 \text{ for all } x \in A,$$

   and

   $$f(x) = 0 \text{ if and only if } x \in \{-1,1\}.$$

   The graph of f intersects the x-axis at $(-1,0)$, $(1,0)$, the y-axis at $(0,\frac{1}{2})$.

   Limits at the extreme points of the domain:

   $$\lim_{x\to\mp\infty} f(x) = +\infty,$$

   $$\lim_{x\to(-2)^-} f(x) = \lim_{x\to 2^+} f(x) = +\infty, \qquad (x=-2,\ x=2 \text{ vertical asymptotes}).$$

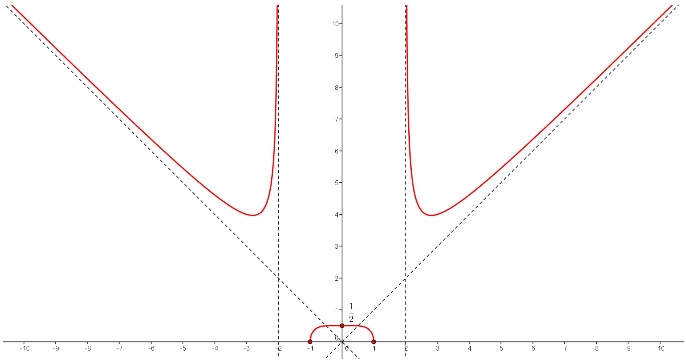
   Moreover,

   $$\lim_{x\to-\infty} \frac{f(x)}{x} = -1, \qquad \lim_{x\to-\infty} \Big(f(x)+x\Big) = 0,$$

   $$\lim_{x\to+\infty} \frac{f(x)}{x} = 1, \qquad \lim_{x\to+\infty} \Big(f(x)-x\Big) = 0,$$

   therefore $y=-x$, $y=x$ are oblique asymptotes as $x\to-\infty$, $x\to+\infty$, respectively.

   The graph of f is the following.



2) The *domain* of g is $A =]-\infty, -1[\cup [1, +\infty[$.

There are *no symmetries* of the graph of g (the function is neither even, nor odd).

Sign and intersection with the axes:

$$g(x) \geq 0 \text{ if and only if } x \in]-\infty, -2] \cup [2, +\infty[,$$

and

$$g(x) = 0 \text{ if and only if } x \in \{-2, 2\}.$$

The graph of g intersects the x-axis at $(-2, 0)$, $(2, 0)$, it does not intersect the y-axis.

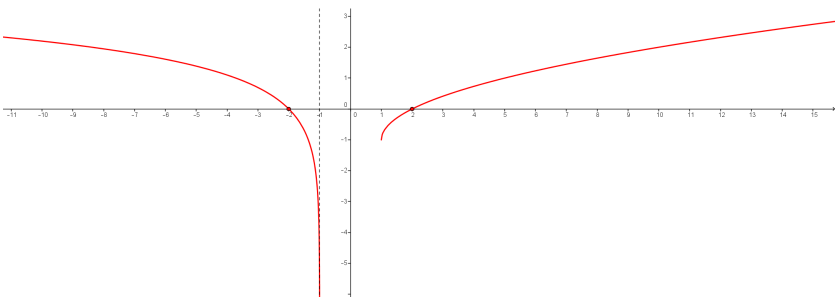
Limits at the extreme points of the domain:

$$\begin{aligned} \lim_{x\to-\infty} g(x) &= +\infty, \\ \lim_{x\to(-1)^-} g(x) &= -\infty \qquad\qquad (x = -1 \text{ vertical asymptote}), \\ g(1) = \lim_{x\to1+} g(x) &= -1, \\ \lim_{x\to+\infty} g(x) &= +\infty, \end{aligned}$$

with

$$\lim_{x\to\mp\infty} \frac{g(x)}{x} = 0,$$

then there is no oblique asymptote as $x \to \mp\infty$.
The graph of g is the following.



Exercise 6. Consider the following functions:

$$g(x) = \frac{2x+3}{(x-2)^2}, \qquad h(x) = x(\log x)^2.$$

For each function, determine the domain, compute the infimum and the supremum and establish if they are the minimum and the maximum, respectively; finally, determine the intervals of convexity, concavity and possible inflection points.

Solutions

- The *domain* of g is $D =]-\infty, 2[\cup]2, +\infty[$.

  Maxima and Minima:
  $g'(x) = -\dfrac{2(x+5)}{(x-2)^3}$;
  $g'(x) = 0 \iff x = -5$, while $g'(x) > 0 \iff x \in]-5, 2[$. Hence, $g(x)$ is increasing for $x \in]-5, 2[$ and it is decreasing for $x \in]-\infty, -5[\cup]2, +\infty[$.
  Then $x = -5$ is a minimum point and $g(-5) = -\frac{1}{7}$ is a global minimum $\Longrightarrow \inf(g) = \min(g) = -\frac{1}{7}$. Moreover $\sup(g) = +\infty$ as

  $$\lim_{x\to 2^-} g(x) = +\infty, \qquad \lim_{x\to 2^+} g(x) = +\infty$$

  Inflexion points:
  $g''(x) = \dfrac{4x+34}{(x-2)^4}$;
  $g''(x) > 0 \iff x \in]-\frac{17}{2}, 2[\cup]2, +\infty[$. Hence g is convex for $x \in]-\frac{17}{2}, 2[\cup]2, +\infty[$ and it is concave for $x \in]-\infty, \frac{17}{2}[$.
  $g''(x) = 0 \iff x = -\frac{17}{2}$ which is an inflexion point.

- The *domain* of h is $D =]0, +\infty[$.

  Maxima and Minima:
  $h'(x) = \log x(\log(x) + 2)$;
  $h'(x) = 0 \iff x \in \left\{\frac{1}{e^2}, 1\right\}$, while $g'(x) > 0 \iff x \in \left]0, \frac{1}{e^2}\right[\cup]1, +\infty[$. Hence, $g(x)$ is increasing for $x \in \left]0, \frac{1}{e^2}\right[\cup]1, +\infty[$ and it is decreasing for $x \in \left]\frac{1}{e^2}, 1\right[$.
  Then, $x = \frac{1}{e^2}$ is a maximum point and $g(\frac{1}{e^2}) = \frac{4}{e^2}$ is a relative maximum as $\lim_{x\to+\infty} h(x) = +\infty$; $\sup(h) = +\infty$.
  $x = 1$ is a minimum point and $h(1) = 0$ is a global minimum, so $\inf(h) = \min(h) = 0$.

  Inflexion points:
  $h''(x) = \dfrac{2}{x}(\log(x) + 1)$;
  $h''(x) > 0 \iff x > \frac{1}{e}$. Hence h is concave for $x \in \left]0, \frac{1}{e}\right[$ and it is convex for $x \in \left]\frac{1}{e}, +\infty\right[$.
  $h''(x) = 0 \iff x = \frac{1}{e}$ which is an inflexion point.

Exercise 8. Study the following functions and draw their graph.

$$1)\ f(x) = \frac{\log(x)+2}{x}$$
$$2)\ f(x) = e^{8x-x^2}$$
$$3)\ f(x) = \frac{2x+1}{\sqrt{x}}$$

Solutions

1) The *domain* of f is $A =]0,+\infty[$.

There are *no symmetries* of the graph of f (the function is neither even, nor odd).

Sign and intersection with the axes:

$$f(x) \geq 0 \text{ if and only if } x \in [\frac{1}{e^2}, +\infty[,$$

and

$$f(x) = 0 \text{ if and only if } x = \frac{1}{e^2}.$$

The graph of f intersects the x-axis at $(\frac{1}{e^2}, 0)$, it does not intersect the y-axis.

Limits at the extreme points of the domain:

$$\lim_{x\to 0^+} f(x) = -\infty, \qquad (x = 0^+ \text{ vertical asymptote})$$
$$\lim_{x\to +\infty} f(x) = 0 \qquad (y = 0 \text{ horizontal asymptote})$$

Maxima and Minima:

$f'(x) = -\dfrac{\log(x)+1}{x^2}$;

$f'(x) = 0 \iff x = \frac{1}{e}$, while $f'(x) > 0 \iff x \in]0, \frac{1}{e}[$. Hence, $f(x)$ is increasing for $x \in]0, \frac{1}{e}[$ and it is decreasing for $x \in]\frac{1}{e}, +\infty[$. Then $x = \frac{1}{e}$ is a maximum point and $f(\frac{1}{e}) = e$ is a global maximum.

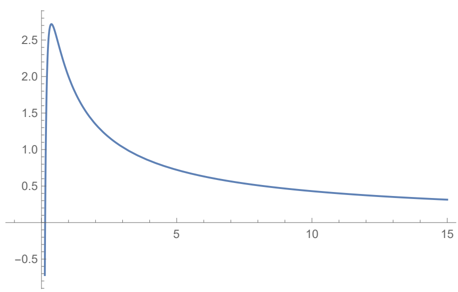
Inflexion points:

$f''(x) = \dfrac{1+2\log(x)}{x^3}$;

$f''(x) > 0 \iff x \in]\frac{1}{\sqrt{e}}, +\infty[$. Hence f is convex for $x \in]\frac{1}{\sqrt{e}}, +\infty[$ and it is concave for $x \in]0, \frac{1}{\sqrt{e}}[$.

$f''(x) = 0 \iff x = \frac{1}{\sqrt{e}}$ which is an inflexion point and $f(\frac{1}{\sqrt{e}}) = \frac{3}{2}\sqrt{e}$.

The graph of f is the following.



2) The *domain* of f is $A = \mathbb{R}$.

There are *no symmetries* of the graph of f (the function is neither even, nor odd).

Sign and intersection with the axes:

$$f(x) > 0 \; \forall \, x \in A.$$

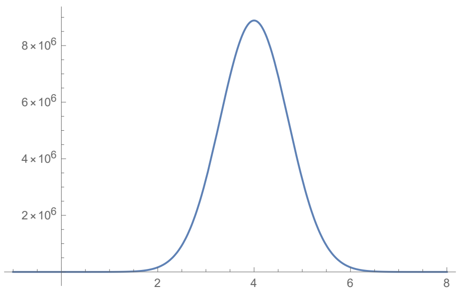
The graph of f intersects the y-axis at $(0,1)$, it does not intersect the x-axis.

Limits at the extreme points of the domain:

$$\lim_{x \to \mp\infty} f(x) = 0, \qquad (y = 0 \text{ horizontal asymptote}).$$

Maxima and Minima:
$f'(x) = e^{8x-x^2} \cdot (8-2x)$;
$f'(x) = 0 \iff x = 4$, while $f'(x) > 0 \iff x < 4$. Hence, $f(x)$ is increasing for $x \in]-\infty, 4[$ and it is decreasing for $x \in]4, +\infty[$. Then $x = 4$ is a maximum point and $f(4) = e^{16}$ is a global maximum.

Inflexion points:
$f''(x) = e^{8x-x^2}(4x^2 - 32x + 62)$;
$f''(x) > 0 \iff x \in \left] -\infty, \frac{8-\sqrt{2}}{2} \right[\cup \left] \frac{8+\sqrt{2}}{2}, +\infty \right[$.
Hence f is convex for $x \in \left] -\infty, \frac{8-\sqrt{2}}{2} \right[\cup \left] \frac{8+\sqrt{2}}{2}, +\infty \right[$ and it is concave for $x \in \left] \frac{8-\sqrt{2}}{2}, \frac{8+\sqrt{2}}{2} \right[$
$f''(x) = 0 \iff x \in \left\{ \frac{8-\sqrt{2}}{2}, \frac{8+\sqrt{2}}{2} \right\}$ which are the inflexion points and $f\left(\frac{8-\sqrt{2}}{2}\right) = f\left(\frac{8+\sqrt{2}}{2}\right) = e^{\frac{31}{2}}$.
The graph of f is the following.



3) The *domain* of f is $A =]0, +\infty[$.

There are *no symmetries* of the graph of f.

Sign and intersection with the axes:

$$f(x) > 0 \; \forall \, x \in D.$$

The graph of f does not intersect the axes.

Limits at the extreme points of the domain:

$$\lim_{x \to 0^+} f(x) = +\infty, \qquad (x = 0^+ \text{vertical asymptote})$$

$$\lim_{x \to +\infty} f(x) = +\infty, \quad \lim_{x \to +\infty} \frac{f(x)}{x} = 0 \qquad (\text{no oblique asymptote})$$

Maxima and Minima:

$f'(x) = \dfrac{2x-1}{2x\sqrt{x}}$;

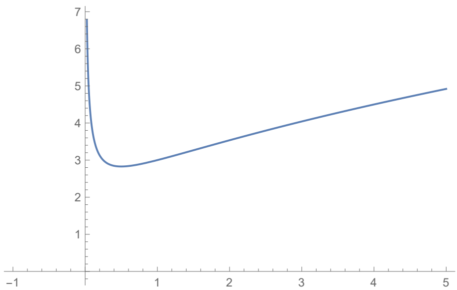
$f'(x) = 0 \iff x = \frac{1}{2}$, while $f'(x) > 0 \iff x > \frac{1}{2}$. Hence, $f(x)$ is increasing for $x \in]\frac{1}{2}, +\infty[$ and it is decreasing for $x \in]0, \frac{1}{2}[$. Then $x = \frac{1}{2}$ is a minimum point and $f(\frac{1}{2}) = 2\sqrt{2}$ is a global minimum.

Inflexion points:

$f''(x) = \dfrac{3-2x}{4\sqrt{x^5}}$;

$f''(x) > 0 \iff x \in \left]0, \frac{3}{2}\right[$. Hence f is convex for $x \in \left]0, \frac{3}{2}\right[$ and it is concave for $x \in \left]\frac{3}{2}, +\infty\right[$

$f''(x) = 0 \iff x = \frac{3}{2}$ which is an inflexion point.
The graph of f is the following.



Exercise 8. Compute the following indefinite integrals, by direct integration.

1) $\displaystyle\int \left(3 + \frac{x}{2} - 2x^3\right) dx$

2) $\displaystyle\int (3x+2)^{-1/2}\, dx$

3) $\displaystyle\int \left(\sin(2x) + 3\cos(2x+1) + e^{3x}\right) dx$

4) $\displaystyle\int \frac{1}{1+x}\, dx$

5) $\displaystyle\int \frac{1+x}{\sqrt{x}}\, dx$

6) $\displaystyle\int \frac{x-1}{\sqrt{1-x^2}}\, dx$

Solutions
In the following the letter c stands for any real constant.

1) $3x + \dfrac{x^2}{4} - \dfrac{x^4}{2} + c$

2) $\dfrac{2}{3}\sqrt{3x+2} + c$

3) $-\dfrac{1}{2}\cos(2x) + \dfrac{3}{2}\sin(2x+1) + \dfrac{1}{3}e^{3x} + c$

4) $\log|x+1| + c$

5) $2\sqrt{x}\left(1 + \dfrac{x}{3}\right) + c$

6) $-\left(\sqrt{1-x^2} + \arcsin x\right) + c$

Exercise 9. Compute the following indefinite integrals, applying when necessary the integration by parts formula or the integration by substitution.

1) $\displaystyle\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx$ 2) $\displaystyle\int x^2 e^x\,dx$

3) $\displaystyle\int \frac{\arctan^2 x}{1+x^2}\,dx$ 4) $\displaystyle\int x\cos(2x)\,dx$

5) $\displaystyle\int \log^2 x\,dx$ 6) $\displaystyle\int e^{\sqrt{x}}\,dx$

7) $\displaystyle\int x^2 \log x\,dx$ 8) $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$

9) $\displaystyle\int \frac{1}{\sqrt{e^x}}\,dx$ 10) $\displaystyle\int \frac{e^{\sqrt{x}}\sin\sqrt{x}}{\sqrt{x}}\,dx$

Solutions

1) $-\arcsin x \cdot \sqrt{1-x^2} + x + c$ 2) $e^x(x^2-2x+2)+c$

3) $\frac{1}{3}\arctan^3 x + c$ 4) $\frac{1}{2}\Big(x\sin(2x) + \frac{1}{2}\cos(2x)\Big) + c$

5) (By integration by parts)

$$\int \log^2 x\,dx = x\log^2 x - 2\int \ln x\,dx = x(\log^2 x - 2\log x + 2) + c$$

6) (By substitution) Applying the substitution

$$u = \sqrt{x}, \quad \text{then} \quad du = \frac{1}{2\sqrt{x}}\,dx, \quad \text{that is} \quad dx = 2u\,du$$

we obtain, by integration by parts,

$$\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du = 2e^u(u-1)+c = 2e^{\sqrt{x}}(\sqrt{x}-1)+c$$

7) (By integration by parts)

$$\int x^2\log x\,dx = \tfrac{x^3}{3}\ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{x^3}{3}\Big(\log x - \tfrac{1}{3}\Big) + c$$

8) (By substitution) Applying the substitution

$$u = \sqrt{x-1}, \qquad \text{that is} \quad x = u^2+1,$$

$$du = \frac{1}{2\sqrt{x-1}}\,dx, \qquad \text{that is} \quad dx = 2u\,du,$$

we have

$$\int \frac{1}{x\sqrt{x-1}}\,dx = \int \frac{2u}{(u^2+1)u}\,du = \int \frac{2}{u^2+1}\,du$$

$$= 2\arctan u + c = 2\arctan\sqrt{x-1} + c$$

9) (By substitution) Applying the substitution

$$u = e^x, \quad \text{then} \quad du = e^x\,dx,$$

we have

$$\int \frac{1}{\sqrt{e^x}}\,dx = \int \frac{1}{u\sqrt{u}}\,du = \int u^{-\frac{3}{2}}\,du = -2u^{-\frac{1}{2}} + c = -2e^{-\frac{x}{2}} + c$$

10) (By substitution and by integration by parts) Notice that, by substitution,

$$u = \sqrt{x}, \quad \text{then} \quad du = \frac{1}{2\sqrt{x}}\, dx,$$

we have

$$\int \frac{e^{\sqrt{x}} \sin \sqrt{x}}{\sqrt{x}}\, dx = 2 \int e^u \sin u\, du,$$

with

$$2 \int e^u \sin u\, du = 2e^u \sin u - 2e^u \cos u - 2 \int e^u \sin u\, du,$$

then

$$2 \int e^u \sin u\, du = e^u (\sin u - \cos u) + c$$

and finally

$$\int \frac{e^{\sqrt{x}} \sin \sqrt{x}}{\sqrt{x}}\, dx = e^{\sqrt{x}} \Big(\sin(\sqrt{x}) - \cos(\sqrt{x}) \Big) + c$$

Exercise 10. Compute the following definite integrals.

$$1) \int_0^1 x e^{-x}\, dx \qquad\qquad 2) \int_0^{\frac{\pi}{2}} \sin x e^{\cos x}\, dx$$

Solutions

1) (By integration by parts)

$$\int_0^1 x e^{-x}\, dx = (-x \cdot e^{-x})\Big|_0^1 - \int_0^1 -e^{-x}\, dx = -e^{-1} - e^{-x}\Big|_0^1 = -2e^{-1} + 1$$

2)

$$\int_0^{\frac{\pi}{2}} \sin x e^{\cos x}\, dx = -e^{\cos x}\Big|_0^{\frac{\pi}{2}} = -(e^{\cos \frac{\pi}{2}} - e^{\cos 0}) = e - 1$$

Exercise 11. Compute the area between the graph of $f(x) = x \sin(2x)$ and the x-axis for $x \in [0, \pi]$.

Solution

$$\text{Area}_{\{f,[0,\pi]\}} = \left| \int_0^{\pi} x \sin(2x)\, dx \right| = \left| \left[\frac{1}{4} \sin(2x) - \frac{x}{2} \cos(2x) \right]_0^{\pi} \right| = \frac{\pi}{2}$$