21 dicembre

<u>Teorema</u> 1) $\frac{\sin x}{x} \in L[0,+\infty]$

2) $\frac{|\sin x|}{x} \notin L[0,+\infty)$

<u>Dim</u> 1) già dimostrato

2) Per ragioni che seguono dalla dimostrazione, verificheremo che $\frac{|\sin x|}{x} \notin L[\pi,+\infty)$.

Supponiamo per assurdo che $\frac{|\sin x|}{x} \in L[\pi,+\infty)$

$$+\infty > \int_{\pi}^{+\infty} \frac{|\sin x|}{x}\,dx = \lim_{y\to+\infty} \int_{\pi}^{y} \frac{|\sin x|}{x}\,dx$$

$$= \lim_{n\to+\infty} \int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx$$

$$= \lim_{n\to+\infty} \sum_{j=1}^{n} \int_{j\pi}^{(j+1)\pi} |\sin x| \, \frac{1}{x}\,dx$$

$$j\pi \le x \le (j+1)\pi \qquad \frac{1}{j\pi} \ge \frac{1}{x} \ge \frac{1}{(j+1)\pi}$$

$$\ge \lim_{n\to+\infty} \sum_{j=1}^{n} \frac{1}{(j+1)\pi} \int_{j\pi}^{(j+1)\pi} |\sin x|\,dx$$

$$x = j\pi + y$$
$$dx = dy$$

$$= \lim_{n\to+\infty} \sum_{j=1}^{n} \frac{1}{(j+1)\pi} \int_{0}^{\pi} |\sin(y + \cancel{j\pi})|\,dy$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \sin y\,dy \lim_{n\to+\infty} \sum_{j=1}^{n} \frac{1}{j+1}$$

$$= \frac{1}{\pi} \left.-\cos y\right]_{0}^{\pi} \lim_{n\to+\infty} \sum_{j=1}^{n} \frac{1}{j+1}$$

$$-\cos\pi + \cos(0) = 2$$

$$\lim_{y\to+\infty}\int_{\pi}^{y}\frac{|\sin x|}{x}\,dx \geq$$

$$\geq \frac{2}{\pi}\lim_{n\to+\infty}\sum_{j=1}^{n}\frac{1}{j+1}$$

$k=j+1$

$$= \frac{2}{\pi}\lim_{n\to+\infty}\sum_{k=2}^{n+1}\frac{1}{k}$$

Notare che $\displaystyle\int_{2}^{n+2}\frac{1}{[x]}\,dx =$

$$= \sum_{k=2}^{n+1}\int_{k}^{k+1}\frac{1}{[x]}\,dx = \sum_{k=2}^{n+1}\int_{k}^{k+1}\frac{1}{k}\,dx$$

$$= \sum_{k=2}^{n+1}\frac{1}{k}$$

$$\lim_{y\to+\infty}\int_{\pi}^{y}\frac{|\sin x|}{x}\,dx \geq \frac{2}{\pi}\lim_{n\to+\infty}\sum_{k=2}^{n+1}\frac{1}{k}$$

$$= \frac{2}{\pi}\lim_{n\to+\infty}\int_{2}^{n+2}\frac{1}{[x]}\,dx = +\infty$$

quest'ultima uguaglianza, segue da $\displaystyle\frac{1}{[x]} \notin L[2,+\infty)$

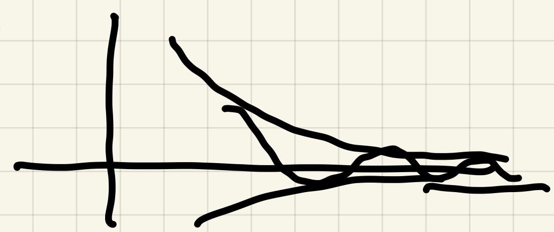
$$\frac{1}{[x]^p} \in L[2,+\infty) \iff p>1$$

E_1 1) $\dfrac{\sin x}{x^p} \in L[1,+\infty) \quad \forall \; p>0$

E' interessante confrontare col fatto che $\dfrac{1}{x^p} \in L[1,+\infty) \iff p>1.$

2) $\dfrac{|\sin x|}{x^p} \in L[1,+\infty) \iff p>1$

Dimostriamo 1 $\qquad p>0$



$$\int_1^y \frac{\sin x}{x^p}\,dx =$$

$$= \int_1^y \frac{(-\cos x)'}{x^p}\,dx = -\frac{\cos x}{x^p}\Big]_1^y$$

$$+ \int_1^y \cos x \,(x^{-p})'\,dx =$$

$$= \cos(1) - \frac{\cos y}{y^p} - p\int_1^y \frac{\cos x}{x^{p+1}}\,dx$$

$\dfrac{\cos x}{x^{p+1}} \in L[1,+\infty)$ perché $\dfrac{|\cos x|}{x^{p+1}} \in L[1,+\infty)$

e questo segue da $\quad 0 \leq \dfrac{|\cos x|}{x^{p+1}} \leq \dfrac{1}{x^{p+1}}$

$p+1>1$

$$\lim_{y\to+\infty}\int_1^y \frac{\sin x}{x^p}\,dx = \cos 1 - p\int_1^{+\infty}\frac{\cos x}{x^{p+1}}\,dx \in \mathbb{R}$$

$$\int_1^{+\infty} \overbrace{\lg\left(1+\frac{1}{x}\right)\operatorname{th}(x)\sin(x)}^{f(x)}\,dx$$

$$f(x) = \lg\left(1+\frac{1}{x}\right)\sin(x) + \lg\left(1+\frac{1}{x}\right)\sin(x)\left(\operatorname{th}(x)-1\right)$$

So che per $x\to+\infty$, $\operatorname{th}(x)-1 = o(x^{-2})$

$\Rightarrow$ Se applico a

$$\left|\lg\left(1+\frac{1}{x}\right)\sin(x)\left(\operatorname{th}(x)-1\right)\right|$$

il confronto asintotico

$$\lim_{x\to+\infty}\frac{\left|\lg\left(1+\frac{1}{x}\right)\sin(x)\left(\operatorname{th}(x)-1\right)\right|}{x^{-2}} = 0$$

$$\Rightarrow \left|\lg\left(1+\frac{1}{x}\right)\sin(x)\left(\operatorname{th}(x)-1\right)\right| \in L[1,+\infty)$$

$$f(x) = \underbrace{\lg\left(1+\frac{1}{x}\right)\sin(x)}_{g(x)} + \underbrace{\lg\left(1+\frac{1}{x}\right)\sin(x)\left(\operatorname{th}(x)-1\right)}_{\text{e' ass. integrabile}}$$

$$\lg(1+y) = y - \frac{y^2}{2} + o(y^2) \qquad y \to 0$$

$$\lg\left(1+\frac{1}{x}\right) = x^{-1} - \frac{x^{-2}}{2} + o(x^{-2}) \qquad x \to +\infty$$

$$g(x) = \left(x^{-1} - \frac{x^{-2}}{2} + o(x^{-2})\right)\sin x =$$

$$= \frac{\sin x}{x} - \frac{1}{2}\,\frac{\sin x}{x^2} + \sin x\, o(x^{-2})$$

$L[1,+\infty)$

e' integrabile ed anche ass. integrabile in $[1,+\infty)$

$$f(x) = \int_0^x \underbrace{e^{-\frac{1}{t}}(t+1)}_{g(t)}\, dt$$

1) Dominio di definizione

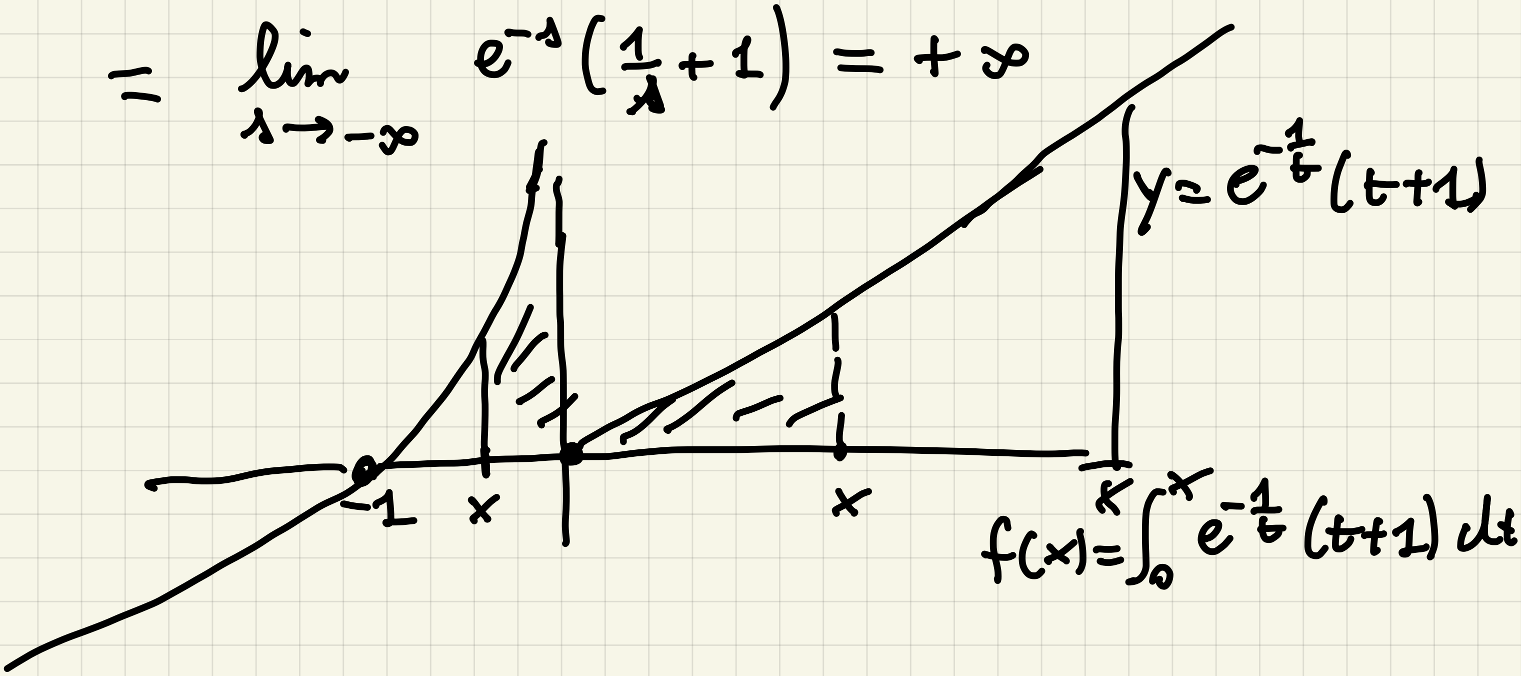
Studiamo il grafico di $y = g(t) = e^{-\frac{1}{t}}(t+1)$

$$\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+} e^{-\frac{1}{t}}(t+1) \qquad s = \frac{1}{t}$$

$$= \lim_{s\to +\infty} e^{-s}\left(\frac{1}{s}+1\right) = \lim_{s\to+\infty} \frac{\frac{1}{s}+1}{e^{s}} = 0$$

$$\lim_{t\to 0^-} g(t) = \lim_{t\to 0^-} e^{-\frac{1}{t}}(t+1) \qquad s = \frac{1}{t}$$

$$= \lim_{s\to -\infty} e^{-s}\left(\frac{1}{s}+1\right) = +\infty$$



$f(x)$ è definita per $x<0$ solo se

$g(t) = e^{-\frac{1}{t}}(t+1) \in L[x,0)$

t^{-1}

$$\lim_{t\to 0^-} \frac{g(t)}{t^{-1}} = \lim_{t\to 0^-} t\, e^{-\frac{1}{t}} \qquad s=\frac{1}{t}$$

$$= \lim_{s\to -\infty} \frac{e^{-s}}{s} = -\infty$$

$$\Longrightarrow g \notin L[x,0) \quad \forall \quad x<0$$

Conclusione: $\mathrm{Dom}\, f = [0, +\infty)$

$$f(x) = \int_0^x e^{-\frac{1}{t}}\,(t+1)\,dt$$

2) $\lim_{x\to+\infty} f(x) = +\infty$ poiché:

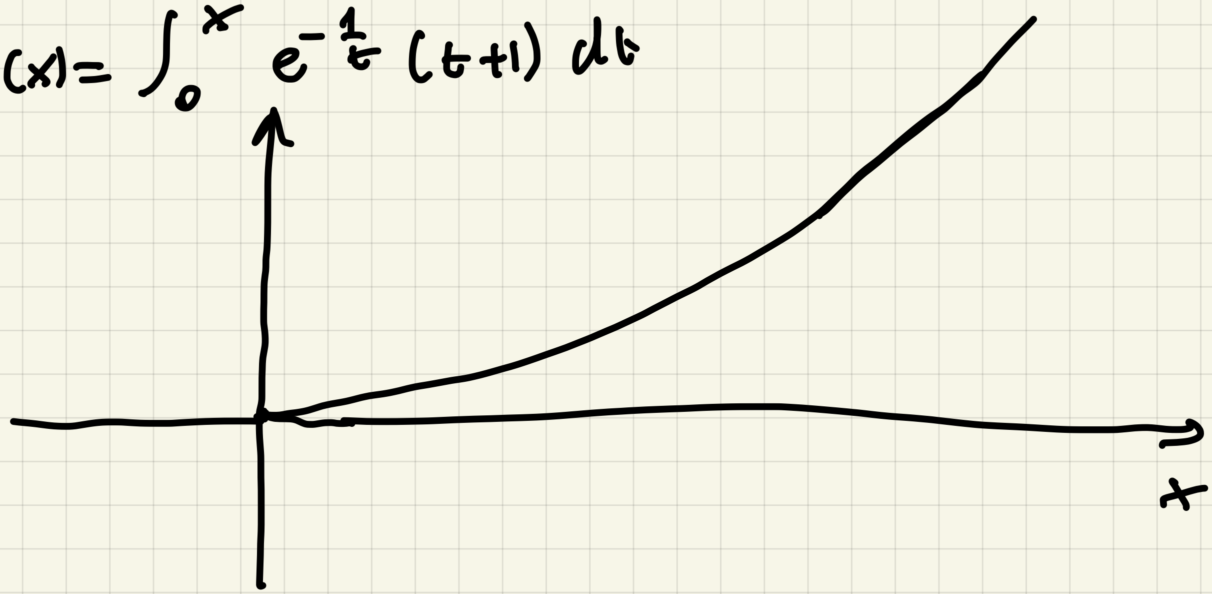
$$\lim_{t\to+\infty} e^{-\frac{1}{t}}(t+1) = +\infty$$

3) $$f'(x) = \begin{cases} e^{-\frac{1}{x}}(x+1) & \text{se } x>0 \\ 0 & \text{se } x=0 \end{cases}$$

4) $$f''(x) = e^{-\frac{1}{x}}\left[1 + (x+1)\,\frac{1}{x^2}\right] =$$

$$= \frac{e^{-\frac{1}{x}}}{x^2}\,\left(x^2+x+1\right) \geq 0$$

$$f(x) = \int_0^x e^{-\frac{1}{t}}(t+1)\,dt$$



$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\int_0^x e^{-\frac{1}{t}}(t+1)\,dt}{x} =$$

$$= \lim_{x \to +\infty} e^{-\frac{1}{x}}(x+1) = +\infty$$

$$g(x) = e^{-\frac{1}{x}}(x+1)$$

$$\lim_{x \to \infty} \frac{g(x)}{x} = \lim_{x \to \infty} e^{-\frac{1}{x}} \frac{x+1}{x} = 1 = m$$

$$\lim_{x \to +\infty} (g(x) - x) = \lim_{x \to +\infty} \left[e^{-\frac{1}{x}} x + e^{-\frac{1}{x}} - x \right]$$

$$= \lim_{x \to +\infty} x\left(e^{-\frac{1}{x}} - 1\right) + \underbrace{\lim_{x \to +\infty} e^{-\frac{1}{x}}}_{1}$$

$$\lim_{x\to+\infty}(g(x)-x)=\lim_{x\to+\infty}\left[e^{-\frac{1}{x}}x+e^{-\frac{1}{x}}-x\right]$$

$$=\underbrace{\lim_{x\to+\infty}x\left(e^{-\frac{1}{x}}-1\right)}_{-1}+\underbrace{\lim_{x\to+\infty}e^{-\frac{1}{x}}}_{1}=0$$

$$\lim_{x\to+\infty}\frac{e^{-\frac{1}{x}}-1}{\frac{1}{x}}=\lim_{y\to0^+}\frac{e^{-y}-1}{y}=\left(e^{-y}\right)'(0)$$

$$=-1$$

$y=x$ è la retta asintotica.