Lemma 13.8. *There exists a constant $C_1 > 1$ and for any fixed $(s,a) \in \mathbb{R}^4$ a sequence $\phi_n \in C_c^\infty\left(\left(s-1/9, s+2^{-(n+1)}\right) \times B_{2^{-1}}(a)\right)$ such that for all $n \geq 2$ we have the following facts:*

(i) $C_1^{-1}2^n \leq \phi_n \leq C_1 2^n$ *and* $|\nabla\phi_n| \leq C_1 2^{2n}$ *in* $Q_{2^{-n}}(s,a)$;

(ii) $\phi_n \leq C_1 2^{-2n}2^{3k}$ *and* $|\nabla\phi_n| \leq C_1 2^{-2n}2^{4k}$ *in* $Q_{2^{-(k-1)}}(s,a) \backslash Q_{2^{-k}}(s,a)$;

(iii) $supp\phi_n \cap ((-\infty, s] \times \mathbb{R}^3) \subset \overline{Q_{1/3}(s,a)}$;

(iv) $|(\partial_t + \triangle)\phi_n| \leq C_1 2^{-2n}$ *in* $(-\infty, s] \times \mathbb{R}^3$.

$$\phi_n(t,x) = 2^{-2n}\vartheta_n(t,x) = 2^{-2n}\chi_n(t,x)\,\Psi_n(t,x)$$

$$\Psi_n(t,x) = \left(4\pi\left(2^{-2n}-t\right)\right)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}}$$

$$\begin{cases} (\partial_t+\Delta)\Psi_n = 0 & \text{for } t < 2^{-2n} \\ \Psi_n\big|_{t=2^{-2n}} = \delta(x) \end{cases}$$

$$\nabla\Psi_n(t,x) = -\pi^{-\frac{3}{2}}\left(4\left(2^{-2n}-t\right)\right)^{-\frac{5}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}}\, x$$

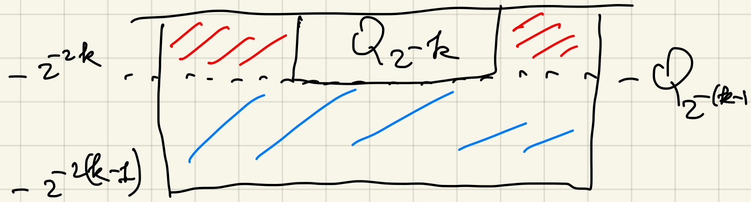
In $Q_{2^{-n}}(0,0)$

$$|\nabla\Psi_n(t,x)| = \pi^{-\frac{3}{2}}\left(4\,2^{-2n}\right)^{-\frac{5}{2}} 2^{-n} =$$

$$= 2^{-5}\pi^{-\frac{3}{2}}\,2^{5n}\,2^{-n} \sim 2^{4n}$$

$$\nabla \varphi_n(t,x) = -\pi^{-\frac{3}{2}} \left(4\left(2^{-2n}+|t|\right)\right)^{-\frac{5}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}} \, x$$

$I_n \qquad Q_{2^{-(k-1)}} \setminus Q_{2^{-k}}$



$$|\nabla \varphi_n(t,x)| \leq \pi^{-\frac{3}{2}} \left(4\left(2^{-2n}+|t|\right)\right)^{-2} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}} \frac{|x|}{\sqrt{4(2^{-2n}+|t|)}}$$

$$\leq \pi^{-\frac{3}{2}} 2^{-4} \left(2^{-2n}+2^{-2k}\right)^{-2} \sup_{\alpha \geq 0} \alpha e^{-\alpha^2}$$

$$\leq \pi^{-\frac{3}{2}} 2^{-4} \, 2^{4k} \sup_{\alpha \geq 0} \alpha e^{-\alpha^2}$$

$$|\nabla \varphi_n(t,x)| \leq \pi^{-\frac{3}{2}} \left(4\left(2^{-2n}+|t|\right)\right)^{-\frac{5}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}} |x|$$

$$\leq 2\pi^{-\frac{3}{2}} \left(4\left(2^{-2n}+|t|\right)\right)^{-\frac{5}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}} 2^{-k}$$

$$= 2\pi^{-\frac{3}{2}} 2^{-k} 2^{+5k} \left(\frac{2^{-2k}}{4(2^{-2n}+|t|)}\right)^{\frac{5}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}}$$

$$\leq 2\pi^{-\frac{3}{2}} 2^{4k} \sup_{\alpha \geq 0} \alpha^{\frac{5}{2}} e^{-\alpha}$$

$$\chi_n(t,x) = X(x)\,T_n(t) \qquad C^\infty_c(\mathbb{R}^4,[0,1])$$

$$X(x) = \begin{cases} 1 & B_{1/4} \\ 0 & B_{1/3} \end{cases}$$

$$T_n(t) = \begin{cases} 1 & t \in (-\frac{1}{16}, 0] \\ 0 & t \leq -\frac{1}{9} \quad t \geq 2^{-2n-1} \end{cases}$$

$T_n|_{(-\infty,0]} = T$ indip. da n

$$\operatorname{supp} \chi_n \cap \overbrace{(\mathbb{R}_- \times \mathbb{R}^3_x)}^{\{t\leq 0\}} = \operatorname{supp}(XT) \cap \overline{Q_1} \subseteq \overline{Q_{1/3}}$$

$$\phi_n = 2^{-2n}\, \overbrace{\chi_n\, \psi_n}^{\vartheta_n}$$

$$\operatorname{supp} \phi_n \cap \{t \leq 0\} \subseteq \overline{Q_{1/3}}$$

$$\nabla\vartheta_n = \psi_n\, \nabla\chi_n + \nabla\psi_n\, \chi_n$$

$Q_{2^{-n}}$

$$|\nabla\vartheta_n| \leq \overbrace{|\psi_n|}^{2^{3n}}\, |\nabla\chi_n| + \overbrace{|\nabla\psi_n|}^{\leq 2^{4n}}$$

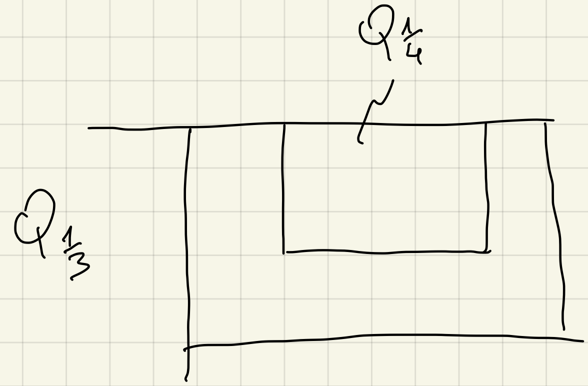
$$|\nabla\chi_n| = T_n\, |\nabla X| \lesssim 1$$

Analogo discorso con le stime in $Q_{2^{-(k-1)}} \setminus Q_{2^{-k}}$

$$(\partial_t+\Delta)\,\phi_n(t,x) = 2^{-2n}(\partial_t+\Delta)(\chi_n\psi_n) =$$

$$= 2^{-2n}(\partial_t+\Delta)\chi_n\,\psi_n + 2\, 2^{-2n}\,\nabla\chi_n\cdot\nabla\psi_n$$

$t\le 0$ $\qquad \chi_n = T(t)\,X(x)$



$$X(x) = \begin{cases} 1 & B_{1/4} \\ 0 & B_{1/3} \end{cases}$$

$$T_n(t) = \begin{cases} 1 & t\in\left(-\frac{1}{16},0\right) \\ 0 & t\le -\frac{1}{9} \quad t\ge 2^{-2n-1} \end{cases}$$

$(\partial_t+\Delta)\chi_n$ ha supporto in $Q_{\frac13}\setminus Q_{\frac14} \subset Q_{2^{-1}}\setminus Q_{2^{-2}}$

$|\psi_n| \lesssim 2^{3k}$ $\qquad k=1$

$$\left|\left(T'X + T\Delta X\right)\right| \lesssim c.$$

In this section we will prove the following theorem.

Theorem 13.2. *There exists absolute constants $\epsilon_0^* > 0$ and $c_M > 0$ s.t. if (u,p) is a suitable weak solution of the NS with*

$$R^{-2}\int_{Q_R(t_0,x_0)} \left(|u|^3 + |p|^{\frac{3}{2}}\right) dtdx < \epsilon_0 \tag{13.2}$$

for an $R>0$ and for a $\epsilon_0 \in (0,\epsilon_0^*]$, then $\|u\|_{L^\infty(Q_{R/2}(t_0,x_0))} \le c_M \epsilon_0^{\frac{1}{3}}$.

$Q_1 = Q_1(0,0)$ $\qquad (s,a) \in Q_{\frac{1}{2}}(0,0)$

$A_n \qquad 2^{2n}\int_{Q_{2^{-n}}(s,a)} |u|^3 + 2^{\frac{3}{2}n}\int_{Q_{2^{-n}}(s,a)} \left|p - (p)_{B_{2^{-n}}(a)}\right|^{\frac{3}{2}} \le 2^{-3n}\,\epsilon_0^{\frac{2}{3}}$

$$(p)_{B_r(a)} = \frac{1}{|B_r(a)|}\int_{B_r(a)} p(t,x)\,dx$$

$B_n \qquad 2^n \sup_{s-2^{-2n}<t<s} \int_{B_{2^{-n}}(a)} |u(t)|^2 dx + 2^n \int_{Q_{2^{-n}}(s,a)} |\nabla u|^2 \le C_B\, 2^{-2n}\,\epsilon_0^{\frac{2}{3}}$

$A_1, \dots, A_4$

$A_1, \dots, A_n \Rightarrow B_{n+1}$

$B_2 \dots B_n \Rightarrow A_n$

$$A_1, \dots, A_4 \qquad q \geq 1$$

$$\int_{Q_r(1,a)} \left| (p)_{B_r(a)} \right|^q \leq \int_{Q_r(1,a)} (|p|)^q_{B_r(a)} =$$

$$= \int_{1-r^2}^{1} \frac{4\pi}{3} r^3 \, (|p|)^q_{B_r(a)} =$$

$$= \int_{1-r^2}^{1} \frac{4\pi}{3} r^3 \left(\frac{1}{\frac{4\pi}{3} r^3} \int_{B_r(a)} |p| \, dx \right)^q$$

$$\leq \int_{1-r^2}^{1} \cancel{\frac{4\pi}{3} r^3} \frac{1}{\cancel{\frac{4\pi}{3} r^3}} \int_{B_r(a)} |p|^q \, dx = \int_{Q_r(1,a)} |p|^q \, dx$$

$$2^{2n} \int_{Q_{2^{-n}}(1,a)} |u|^3 + 2^{\frac{3}{2}n} \int_{Q_{2^{-n}}(1,a)} \left| p - (p)_{B_{2^{-n}}(a)} \right|^{\frac{3}{2}} \leq$$

$$\leq 2^{2n} \int_{Q_{2^{-n}}(1,a)} |u|^3 + 2^{\frac{3}{2}n} \, 2^{\frac{1}{2}2} \int_{Q_{2^{-n}}(1,a)} |p|^{\frac{3}{2}}$$

$$\leq 2^{\frac{3}{2}} \, 2^{2n} \int_{Q_{2^{-n}}(1,a)} \left(|u|^3 + |p|^{\frac{3}{2}} \right)$$

$$\leq 2^{\frac{3}{2}} \, 2^{2n} \int_{Q_1} \left(|u|^3 + |p|^{\frac{3}{2}} \right) \leq \underbrace{\left(2^{\frac{3}{2}} \, 2^{2n} \, \varepsilon_0^{\frac{1}{3}} \, 2^{3n} \right)}_{< 1} 2^{-3n} \cdot \varepsilon_0^{\frac{2}{3}}$$

$$A_1 \dots A_n \Longrightarrow B_{n+1}$$

$$t \le 0$$

$$\int_{B_1} |u(t)|^2 \phi_n(t) + 2 \int_{-1}^{t} |\nabla u|^2 \phi_n \le$$

$$\le \int_{-1}^{t} \int_{B_1} |u|^2 (\partial_t + \Delta) \phi_n + \int_{-1}^{t} \int_{B_1} (|u|^2 + 2p) u \cdot \nabla \phi_n$$

$$C_1^{-1} 2^n \sup_{1-2^{-2n} < t < 1} \int_{B_{2^{-n}}(u)} |u(t)|^2 + 2 C_1^{-1} 2^n \int_{Q_{2^{-2n}}(1,u)} |\nabla u|^2 \le$$

$$\le \frac{3}{2} C_1 2^{-2n} \int_{Q_{\frac{1}{3}}(1,u)} |u|^2 + \frac{3}{2} \int_{Q_{\frac{1}{3}}(1,u)} |u|^3 |\nabla \phi_n| +$$

$$+ 3 \int_{Q_{\frac{1}{3}}(1,u)} p \, u \cdot \nabla \phi_n$$

$$= I_1 + I_2 + I_3$$

$$I_1 \le C_1 2^{-2n} \varepsilon_0^{2/3} \quad , \quad I_2 \le \frac{3}{2} C_1 2^{-2n} \varepsilon_0^{\frac{2}{3}}$$

$$I_3 = \int_{Q_{\frac{1}{3}}(1,u)} p\, u \cdot \nabla \phi_n$$

$$\chi_k \in C_c^\infty\left((-\infty, s] \times \mathbb{R}^3, [0,1] \right) \qquad k = 1, \dots n$$

$$\chi_k = 1 \quad \text{in} \quad Q_{\frac{7}{8}2^{-k}}(1,u)$$

$$\operatorname{supp} \chi_k \cap \overline{Q_1(1,u)} \subseteq \overline{Q_{2^{-k}}(1,u)}$$

$$|\nabla \chi_k| \leq 2^k \, 16$$

$$= \sum_{k=1}^{n-1} \int_{Q_{\frac{1}{3}}(1,u)} p\, u \cdot \nabla \left[\phi_n (\chi_k - \chi_{k+1}) \right] + \int_{Q_{\frac{1}{3}}(1,u)} p\, u \cdot \nabla \left[\phi_n \chi_n \right]$$

$$\chi_1 \quad \text{in} \quad Q_{\frac{1}{3}}(1,u)$$

$$\chi_1 = 1 \quad \text{in} \quad Q_{\frac{7}{8}2^{-1}}(1,u)$$

$$\frac{7}{16} > \frac{1}{3} \qquad 21 > 16$$

$$1 = \sum_{k=1}^{n-1} \left(\chi_k - \chi_{k+1} \right) + \chi_n$$

$$I_3 = \sum_{k=1}^{n-1} \int_{Q_{\frac{1}{3}}(1,a)} p\, u \cdot \nabla \left[\phi_n (\chi_k - \chi_{k+1}) \right] + \int_{Q_{\frac{1}{3}}(1,a)} p\, u \cdot \nabla [\phi_n \chi_n]$$

$$= \sum_{k=1}^{n-1} \int_{Q_{\frac{1}{3}}(1,a)} \left(p - (p)_{B_{2^{-k}}(a)}\right) u \cdot \nabla \left[\phi_n (\chi_k - \chi_{k+1})\right] + \int_{Q_{\frac{1}{3}}(1,a)} \left(p - (p)_{B_{2^{-n}}(a)}\right) \cdot \nabla [\phi_n \chi_n]$$

$$\operatorname{supp} \chi_k \cap \overline{Q_{\frac{1}{k}}(1,a)} \subseteq \overline{Q_{2^{-k}}(1,a)}$$

$$|I_3| \leq \sum_{k=1}^{n-1} \int_{Q_{2^{-k}}(1,a)} \left|p - (p)_{B_{2^{-k}}(a)}\right| |u| \cdot \left|\nabla \left[\phi_n (\chi_k - \chi_{k+1})\right]\right| + \int_{Q_{2^{-n}}(1,a)} \left|p - (p)_{B_{2^{-n}}(a)}\right| \cdot \left|\nabla [\phi_n \chi_n]\right|$$

$$\left|\nabla \left[\phi_n (\chi_k - \chi_{k+1})\right]\right| \leq |\nabla \phi_n| \,(\chi_k - \chi_{k+1}) \leq$$

$$\leq |\nabla \phi_n| \left(\chi_{Q_{2^{-k}}(1,a) \setminus Q_{2^{-k-1}}(1,a)} + \chi_{Q_{2^{-k-1}}(1,a) \setminus Q_{2^{-k-2}}(1,a)} \right.$$

$$\operatorname{supp} (\chi_k - \chi_{k+1}) \subseteq Q_{2^{-k}}(1,a) \setminus Q_{2^{-k-2}}(1,a)$$

$$\lesssim 2^{-2n}\, 2^{4k}$$

$$|I_3| \le \sum_{k=1}^{n-1} \int_{Q_{2^{-k}}(1,a)} \left|p - (p)_{B_{2^{-k}}(a)}\right| |u| \cdot \left|\nabla\left[\phi_n(\chi_k - \chi_{k+1})\right]\right| + \int_{Q_{2^{-n}}(1,a)} \left|p - (p)_{B_{2^{-n}}(a)}\right| \cdot \left|\nabla\left[\phi_n \chi_n\right]\right|$$

$$4 = \frac{7}{3} + \frac{2}{3} + 1$$

$$\lesssim 2^{-2n} \sum_{k=1}^{n} 2^{4k} \int_{Q_{2^{-k}}(1,a)} \left|p - (p)_{B_{2^{-k}}(a)}\right| |u|$$

$$\le 2^{-2n} \sum_{k=1}^{n} 2^{\frac{7}{3}k} \left(2^{\frac{2}{3}k} \|u\|_{L^3(Q_{2^{-k}})}\right) \left(2^{k} \left\|p - (p)_{B_{2^{-k}}(a)}\right\|_{L^{\frac{3}{2}}}\right)$$

$$\le 2^{-2n} \sum_{k=1}^{n} 2^{\frac{7}{3}k} \underbrace{\left(2^{2k} \int_{Q_{2^{-k}}} |u|^3 + 2^{\frac{3}{2}k} \int_{Q_{2^{-k}}} \left|p - (p)_{B_{2^{-k}}(a)}\right|^{\frac{3}{2}}\right)}_{2^{-3k} \varepsilon_0^{\frac{2}{3}}}$$

$$\le 2^{-2n} \varepsilon_0^{\frac{2}{3}} \sum_{k=1}^{n} 2^{\left(\frac{7}{3} - 3\right)k}$$

$$\le \varepsilon_0^{\frac{2}{3}} 2^{-2n} \sum_{k=1}^{\infty} 2^{-\frac{2}{3}k} \lesssim \varepsilon_0^{\frac{2}{3}} 2^{-2n}$$

$$I_1 + I_2 + I_3 \le c_0 \, 2^{-2n} \varepsilon_0^{\frac{2}{3}}$$

$$C_1^{-1} 2^n \sup_{s-2^{-2n}<t<s} \int_{B_{2^{-n}}(x)} |u(t)|^2 + 2 C_1^{-1} 2^n \int_{Q_{2^{-2n}}(s,x)} |\nabla u|^2 \le$$

$$\le \frac{3}{2} C_1 2^{-2n} \int_{Q_{\frac{1}{3}}(s,x)} |u|^2 + \frac{3}{2} \int_{Q_{\frac{1}{3}}(s,x)} |u|^3 |\nabla \phi_n| +$$

$$+ 3 \int_{Q_{\frac{1}{3}}(s,x)} p\, u \cdot \nabla \phi_n$$

$$= I_1 + I_2 + I_3 < C_0 \varepsilon_0^{\frac{2}{3}} 2^{-2n}$$

$$2^{n+1} \sup_{s-2^{-2n-2}<t<s} \int_{B_{2^{-n-1}}} |u(t)|^2 + 2\, 2^{n+1} \int_{Q_{2^{-(n+1)}}(s,x)} |\nabla u|^2$$

$$< \underbrace{C_1 C_0 2^3}_{C_B} \varepsilon_0^{\frac{2}{3}} 2^{-2n}$$