Sequence

$A \subset \mathbb{N}$ unbounded

$f : A \to \mathbb{R}$ is called a <u>sequence</u>
$\quad n \mapsto f(n)$

Paper November 23, 2020

<u>Exercise 4</u>

7) $\displaystyle \lim_{n \to +\infty} \underbrace{\frac{\sin(n\pi)}{n}}_{= f(n)} = \lim_{n \to +\infty} 0 = 0$

$\mathrm{dom}(f) = \mathbb{N} \setminus \{0\}$

Notice that, for any $n \in \mathbb{N} \setminus \{0\}$, $\sin(n\pi) = 0 \longrightarrow f(n) = \dfrac{\sin(n\pi)}{n} = 0$
$\forall n \in \mathrm{dom}(f)$

8) $\lim_{n \to +\infty} \underbrace{\cos(n^2 \pi)}_{\substack{\shortparallel \\ f(n)}}$ $\quad \nexists$ $\quad dom(f) = \mathbb{N}$

n	n^2	$\cos(n^2\pi)$
0	0	$\cos(0) = 1$
1	$1^2 = 1$	$\cos(\pi) = -1$
2	4	$\cos(4\pi) = \cos(0) = 1$
3	9	$\cos(9\pi) = \cos(\pi) = -1$
4	16	$\cos(16\pi) = \cos(0) = 1$
5	25	$\cos(25\pi) = \cos(\pi) = -1$
$\vdots$	$\vdots$	$\vdots$

$$\cos(n^2\pi) = \begin{cases} \cos(0) = 1 & \text{if } n \text{ is even} \\ \cos(\pi) = -1 & \text{if } n \text{ is odd} \end{cases}$$

$\cos(n^2\pi)$ oscillates between -1 and 1

15) $\lim_{n \to +\infty} \underbrace{(-1)^n}_{= f_1(n)} \underbrace{\left(\cos^2(n\pi) - 1\right)}_{= f_2(n)}$ $\qquad dom(f) = \mathbb{N}$

$$f(n) = f_1(n) \cdot f_2(n)$$

$$f_1(n) = (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

$$f_2(n) = \cos^2(n\pi) - 1 = \begin{cases} 1 - 1 = 0 & \text{if } n \text{ is even} \\ (-1)^2 - 1 = 1 - 1 = 0 & \text{if } n \text{ is odd} \end{cases}$$

$$= 0 \quad \forall n \in \mathbb{N}$$

$$f(n) = (-1)^n \cdot 0 = 0 \quad \forall n \in \mathbb{N} \longrightarrow \lim_{n \to +\infty} f(n) = 0$$

13) $\lim_{n \to +\infty} \underbrace{\left(n - 3\sqrt{n}\right)}_{\substack{\shortparallel \\ f(n)}}$

$dom(f) = \mathbb{N}$

$(I.F. \ +\infty - \infty)$

$$= \lim_{n \to +\infty} \left(n - 3\sqrt{n}\right) \cdot \frac{n + 3\sqrt{n}}{n + 3\sqrt{n}}$$

$$= \lim_{n \to +\infty} \frac{\left(n - 3\sqrt{n}\right) \cdot \left(n + 3\sqrt{n}\right)}{n + 3\sqrt{n}}$$

$(a+b)(a-b) = a^2 - b^2$

$a = n \qquad a^2 = n^2$

$b = 3\sqrt{n} \qquad b^2 = 9n$

$$= \lim_{n \to +\infty} \frac{n^2 - 9n}{n + 3\sqrt{n}}$$

$$= \lim_{n \to +\infty} \frac{n^{\cancel{2}}\left(1 - \frac{9}{n}\right)}{\cancel{n}\left(1 + \frac{3}{\sqrt{n}}\right)} = \lim_{n \to +\infty} n = +\infty$$

$\left(1 - \frac{9}{n}\right) \to 1$

$\left(1 + \frac{3}{\sqrt{n}}\right) \to 1$

Exercise 5

2) $$\lim_{x \to +\infty} \frac{3x}{\underbrace{x-\sqrt{x}}_{= f(x)}} = \lim_{x \to +\infty} \frac{3\cancel{x}}{\cancel{x}\left(1 - \frac{1}{\sqrt{x}}\right)} = 3$$

$$\left(1 - \frac{1}{\sqrt{x}}\right) \to 1$$

4) $$\lim_{x \to 0} \frac{\sin(\pi x)}{3x} =$$

$$\left(\text{I.F. } \frac{0}{0}\right)$$

R.L. $$\lim_{t \to 0} \frac{\sin t}{t} = 1$$

$$\lim_{x \to 0} \frac{\sin(\pi x)}{3x} \cdot \frac{\pi}{\pi} =$$

$$\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} \cdot \frac{\pi}{3} = \frac{\pi}{3} \cdot \lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = \frac{\pi}{3} \cdot \lim_{t \to 0} \frac{\sin t}{t} = \frac{\pi}{3}$$

limit of the compound function

$\boxed{t = \pi x}$ As $x \to 0$, $t \to 0$

10) $\lim_{x \to 0^+} \frac{1 - \cos\sqrt{x}}{x}$

R.L. $\lim_{t \to 0} \frac{1-\cos t}{t^2} = \frac{1}{2}$

$= \lim_{t \to 0^+} \frac{1-\cos t}{t^2} = \frac{1}{2}$

$t = \sqrt{x} \quad t^2 = x$

As $x \to 0^+$, $t \to 0^+$

limit of the compound

11) $\lim_{x \to 1} \frac{\sin(\sqrt{x} - 1)}{x - 1} \overset{(*)}{=}$

R.L. $\lim_{t \to 0} \frac{\sin t}{t} = 1 \quad \left(\text{I.F. } \frac{0}{0}\right)$

Notice $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$

$a^2 - b^2 = (a-b)(a+b)$

$a = \sqrt{x} \quad b = 1$

$a^2 = x \quad b^2 = 1$

(*)

$= \lim_{x \to 1} \frac{\sin(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{\sin(\sqrt{x}-1)}{\sqrt{x}-1} \cdot \frac{1}{\sqrt{x}+1}$

$$f_1(x) = \frac{\sin(\sqrt{x} - 1)}{\sqrt{x} - 1}$$

$$\lim_{x \to 1} f_1(x) = \lim_{x \to 1} \frac{\sin(\sqrt{x} - 1)}{\sqrt{x} - 1}$$

$t = \sqrt{x} - 1$

As $x \to 1$, $t \longrightarrow 0$

$$= \lim_{t \to 0} \frac{\sin t}{t} = 1$$

limit of the compound

$$f_2(x) = \frac{1}{\sqrt{x} + 1}$$

$$\lim_{x \to 1} f_2(x) = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

because f_2 is continuous at $x = 1$ and then

$$\lim_{x \to 1} f_2(x) = f_2(1) = \frac{1}{2}$$

$$\longrightarrow \quad \lim_{x \to 1} f_1(x) \cdot f_2(x) = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Paper December 18, 2020

Exercise 5

b) $$\lim_{x \to -\infty} \frac{x^2 + \sin x}{x^2 + 2x - 5} = \lim_{x \to -\infty} \frac{\cancel{x^2}\left(1 + \frac{\sin x}{x^2}\right)}{\cancel{x^2}\left(1 + \frac{2}{x} - \frac{5}{x^2}\right)} = 1$$

$1 + \frac{\sin x}{x^2} \to 1$, $1 + \frac{2}{x} - \frac{5}{x^2} \to 1$

$$-1 \le \sin x \le 1 \qquad \forall x \in \mathbb{R}$$

$$-\frac{1}{x^2} \le \frac{\sin x}{x^2} \le \frac{1}{x^2} \qquad \forall x \in \mathbb{R} \setminus \{0\}$$

as $x \to -\infty$: $-\frac{1}{x^2} \to 0$, $\frac{\sin x}{x^2} \to 0$, $\frac{1}{x^2} \to 0$

by the squeeze theorem

d) $\lim_{x \to +\infty} \left(x^3 + 2x\right) \cdot 2^{-x} =$

$$2^{-x} = \frac{1}{2^x}$$

$$\lim_{x \to +\infty} \frac{x^3 + 2x}{2^x} =$$

$$\lim_{x \to +\infty} \frac{x^3\left(1 + \frac{2}{x^2}\right)}{2^x} =$$

$$\lim_{x \to +\infty} \underbrace{\frac{x^3}{2^x}}_{0} \cdot \underbrace{\left(1 + \frac{2}{x^2}\right)}_{1} = 0$$

R.L. $\lim_{x \to +\infty} \frac{k^x}{x^r} = +\infty$

$k > 1, \; r > 0$

$(k = 2, \; r = 3)$

$\lim_{x \to +\infty} \frac{x^3}{2^x} = 0$ by the limit of the reciprocal function

$$\lim_{x \to +\infty} \frac{2^x}{x^3} = +\infty$$

e) $\lim\limits_{x \to +\infty} \left(x^2 + 3^x - \log_3 x\right) \cdot 3^{-x} =$

$3^{-x} = \dfrac{1}{3^x}$

$\lim\limits_{x \to +\infty} \dfrac{x^2 + 3^x - \log_3 x}{3^x} =$

$\lim\limits_{x \to +\infty} \dfrac{\cancel{3^x}\left(\dfrac{x^2}{3^x} + 1 - \dfrac{\log_3 x}{3^x}\right)}{\cancel{3^x}} = 1$

$\dfrac{x^2}{3^x} \to 0$, $\dfrac{\log_3 x}{3^x} \to 0$

$(k > 1,\ r > 0)$

R.L. $\lim\limits_{x \to +\infty} \dfrac{k^x}{x^r} = +\infty$

$(k = 3,\ r = 2)$

$\lim\limits_{x \to +\infty} \dfrac{x^2}{3^x} = 0$

by the limit of the reciprocal

$\lim\limits_{x \to +\infty} \dfrac{3^x}{x^2} = +\infty$

$\lim\limits_{x \to +\infty} \dfrac{x^2}{3^x} = 0$

R.L. $\lim\limits_{x \to +\infty} \dfrac{x^r}{\log_k x} = +\infty$

$r > 0,\ k > 1$

$$\lim_{x \to +\infty} \frac{\log_3 x}{3^x} = \lim_{x \to +\infty} \frac{\log_3 x}{3^x} \cdot \frac{x}{x}$$

$$= \lim_{x \to +\infty} \underbrace{\frac{\log_3 x}{x}}_{\to 0} \cdot \underbrace{\frac{x}{3^x}}_{\to 0} = 0$$

by $\lim_{x \to +\infty} \frac{x}{\log_3 x} = +\infty$ by $\lim_{x \to +\infty} \frac{3^x}{x} = +\infty$

f) $\lim_{x \to 0} (1+x)^{\frac{1}{2x}} =$

$$\lim_{x \to 0} \Big[\underbrace{(1+x)^{\frac{1}{x}}}_{\to e} \Big]^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$$

limit of the compound

R.L. $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

(Neper number)

k) $\lim\limits_{x \to 0} \sin x \cdot \log_6 x =$ (I.F. $0 \cdot \infty$)

$\lim\limits_{x \to 0} \sin x \cdot \log_6 x \cdot \frac{x}{x} =$

$\lim\limits_{x \to 0} \underbrace{\frac{\sin x}{x}}_{\downarrow \, 1} \cdot \underbrace{x \log_6 x}_{\downarrow \, 0} = 0$

R.L. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

R.L. $\lim\limits_{x \to 0} x^r \cdot \log_k x = 0$

$r > 0, \; k > 1$

$(r = 1, \; k = 6)$

p) $\lim\limits_{x\to 0} \dfrac{e^{x \sin x} - 1}{1 - \cos x} =$ $\left(\text{I.F. } \dfrac{0}{0}\right)$

R.L. $\lim\limits_{t\to 0} \dfrac{e^{t} - 1}{t} = 1$

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

$\lim\limits_{x\to 0} \dfrac{e^{x \sin x} - 1}{1 - \cos x} \cdot \dfrac{x \sin x}{x \sin x} =$

$\lim\limits_{x\to 0} \underbrace{\dfrac{e^{x \sin x} - 1}{x \sin x}}_{f_1(x)} \cdot \underbrace{\dfrac{x \sin x}{1 - \cos x}}_{f_2(x)} = 1 \cdot 2 = 2$

- $\lim\limits_{x\to 0} \dfrac{e^{x \sin x} - 1}{x \sin x} = \lim\limits_{t\to 0} \dfrac{e^{t} - 1}{t} = 1$

$t = x \sin x$

An $x \to 0$, $t \to 0$

$\lim\limits_{x\to 0} \dfrac{x \sin x}{1 - \cos x} =$

$\lim\limits_{x\to 0} \dfrac{x \sin x}{1 - \cos x} \cdot \dfrac{x}{x} =$

$\lim\limits_{x\to 0} \underbrace{\dfrac{x^2}{1 - \cos x}}_{\downarrow\, 2} \cdot \underbrace{\dfrac{\sin x}{x}}_{\downarrow\, 1} = 2$

r) $\lim_{x \to 0} \frac{\log(\cos x)}{x^2} \underset{(*)}{=}$ $\left(\text{I.F. } \frac{0}{0}\right)$

$\left(\log = \ln = \log_e\right.$ natural logarithm

logarithm with base

$\left.e = \text{Nepr number}\right)$

$\log(\cos x) = \log(\cos x - 1 + 1)$

R.L. $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$

$$(*) = \lim_{x \to 0} \frac{\log(\cos x - 1 + 1)}{x^2} \cdot \frac{\cos x - 1}{\cos x - 1} =$$

$$= \lim_{x \to 0} \underbrace{\frac{\log(\cos x - 1 + 1)}{\cos x - 1}}_{\substack{\| \\ f_1(x) \\ \downarrow \\ 1}} \cdot \underbrace{\frac{\cos x - 1}{x^2}}_{\substack{\| \\ f_2(x) \\ \downarrow \\ -\frac{1}{2}}} = 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$\lim_{x \to 0} \frac{\log(\cos x - 1 + 1)}{\cos x - 1} = \lim_{t \to 0} \frac{\log(t+1)}{t} = 1$$

limit of compound

R.L. $\lim_{t \to 0} \frac{\log(t+1)}{t} = 1$

$t = \cos x - 1$

As $x \to 0$, $t \to 0$