7 Gennaio

Lezione l'8 } in presenza

Salta la lezione lunedì 11
Martedì l'ultima lezione in remoto
Giov. e Venerdì "ricevimento"
in remoto.

9/7/2018

Al variare di $a>0$ calcolare

$$\lim_{x\to 0^+} \frac{\sin(x^a + x^{2a}) - \sqrt{x}}{\int_0^x \sin\left(\frac{1}{t}\right) dt - x}$$

E' un limite $\frac{0}{0}$.

Hôpital qui non funziona bene

da $\frac{d}{dx}\int_0^x \sin\left(\frac{1}{t}\right) dt = \sin\left(\frac{1}{x}\right)$

$$\lim_{x\to 0^+} \frac{\sin(x^a + x^{2a}) - \sqrt{x}}{\int_0^x \sin\left(\frac{1}{t}\right) dt - x}$$

$$\frac{\cos(x^a + x^{2a})\left(a x^{a-1} + 2a x^{2a-1}\right) - \frac{1}{2} x^{-\frac{1}{2}}}{\sin\left(\frac{1}{x}\right) - 1}$$

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \Longrightarrow \sin\left(\frac{1}{x_n}\right) - 1 =$$

$$= \sin\left(\frac{\pi}{2} + 2\pi n\right) - 1 = 0$$

$$\lim_{x\to 0^+} \frac{\sin(x^a + x^{2a}) - \sqrt{x}}{\int_0^x \sin\left(\frac{1}{t}\right) dt - x}$$

Cerchiamo ~~[illegible]~~ separatamente, termine dominante al numeratore e termine dominante al denominatore.

Denominatore. Poniamo $F(x) = \int_0^x \sin\left(\frac{1}{t}\right) dt$

Sappiamo che $F(0) = 0$.

Osserviamo che $F(x)$ e' una primitiva della funzione $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

$\Longrightarrow F'(0) = 0 \Longrightarrow F(x) = o(x)$

$$F(x) = \underbrace{F(0)}_{0} + \underbrace{F'(0)}_{0}x + o(x) = o(x)$$

Conclusione.

Denominatore $= \int_0^x \sin\left(\frac{1}{t}\right) dt - x = F(x) - x$

$= o(x) - x = -x\,(1 + o(1))$

$$\lim_{x\to 0^+} \frac{\sin(x^a + x^{2a}) - \sqrt{x}}{\int_0^x \sin\left(\frac{1}{t}\right) dt - x} = \lim_{x\to 0^+} \frac{\sin(x^a + x^{2a}) - \sqrt{x}}{-x}$$

$$-\cos(x^a + x^{2a})\left(a\,x^{a-1} + 2a\,x^{2a-1}\right) + \frac{1}{2}x^{-\frac{1}{2}} =: g(x)$$

Chiaramente $a = \frac{1}{2}$ e' un valore "critico" del parametro a.

Per $a > \frac{1}{2}$ si ha $x^{a-1} = o(x^{-\frac{1}{2}})$

$\Rightarrow x^{2a-1} = o(x^{-\frac{1}{2}})$

Quindi per $a > \frac{1}{2}$ ho

$$-\cos(x^a + x^{2a})\left(a\,x^{a-1} + 2a\,x^{2a-1}\right) + \frac{1}{2}x^{-\frac{1}{2}} =$$

$$= \frac{1}{2}x^{-\frac{1}{2}} + o(x^{-\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}(1 + o(1)).$$

$\Rightarrow$ limite $= \lim_{x\to 0^+} \frac{1}{2}x^{-\frac{1}{2}} = +\infty$

Per $0<a<\frac{1}{2}$

$$-\cos(x^a+x^{2a})\left(a x^{a-1}+2a x^{2a-1}\right)+\frac{1}{2}x^{-\frac{1}{2}}=g(x)$$

ho $\quad x^{-\frac{1}{2}}=o(x^{a-1})$, $x^{2a-1}=o(x^{a-1})$,

$\cos(x^a+x^{2a})=1+o(1)$, quindi

$$g(x)=-(1+o(1))\left(a x^{a-1}+o(x^{a-1})\right)+o(x^{a-1})$$

$$=-a x^{a-1}+o(x^{a-1})=-a x^{a-1}(1+o(1))$$

il limite $=\lim_{x\to 0^+}-a x^{a-1}=-\infty$

quando $\quad 0<a<\frac{1}{2}$

Resta da considerare $a=\frac{1}{2}$.

$a = \frac{1}{2}$

$$-\cos(x^a + x^{2a})\left(a x^{a-1} + 2a\, x^{2a-1}\right) + \frac{1}{2} x^{-\frac{1}{2}} = g(x)$$

$$= -\cos(x^{\frac{1}{2}} + x)\left(\frac{1}{2} x^{-\frac{1}{2}} + 1\right) + \frac{1}{2} x^{-\frac{1}{2}} =$$

$$= \frac{1}{2} x^{-\frac{1}{2}} \left(1 - \cos(x^{\frac{1}{2}} + x)\right) - \cos(x^{\frac{1}{2}} + x)$$

Ricordiamo che

$$\lim_{y \to 0} \frac{1-\cos(y)}{y} = \frac{(1-\cos(y))'}{(y)'}(0) = \frac{\sin(0)}{1} = 0$$

$$= \frac{1}{2} \frac{x^{\frac{1}{2}} + x}{x^{\frac{1}{2}}} \quad \frac{1 - \cos(x^{\frac{1}{2}} + x)}{x^{\frac{1}{2}} + x} - 1 + o(1)$$

$$= \underbrace{\frac{1}{2} \left(1 + o(1)\right) \quad o(1)}_{o(1)} - 1 + o(1) = -1 + o(1)$$

Concludo che il nostro limite è

$$\lim_{x \to 0^+} -1 = -1 \qquad (\text{con } a = \tfrac{1}{2})$$

Stabilire per quali $p>0$ si ha $\sin(x^p) \in L([0,+\infty))$

Osserviamo che $\sin(x^p) \in C^0([0,+\infty))$

$\Rightarrow \sin(x^p) \in L([0,1])$

sicché basta stabilire per quali $p>0$ si ha $\sin(x^p) \in L([1,+\infty))$

$$x = y^{\frac{1}{p}}$$

$$y = x^p$$

$$dy = p\, x^{p-1}\, dx = p\, y^{\frac{p-1}{p}}\, dx$$

$$dx = \frac{dy}{p\, y^{1-\frac{1}{p}}}$$

$$\int_1^R \sin(x^p)\, dx = \frac{1}{p}\int_1^{R^p} \frac{\sin(y)}{y^{1-\frac{1}{p}}}\, dy$$

$$\lim_{R\to+\infty} \int_1^R \sin(x^p)\, dx = \frac{1}{p} \lim_{R\to+\infty} \int_1^R \frac{\sin(y)}{y^{1-\frac{1}{p}}}\, dy$$

$$\lim_{R\to+\infty}\int_1^R \frac{\sin(x)}{x^q}\,dx \qquad q = 1-\frac{1}{p}$$

Si ha che $\dfrac{\sin x}{x^q} \in L[1,+\infty)$ quando $q>0$.

$$q>0 \iff 1-\frac{1}{p}>0 \iff 1>\frac{1}{p} \iff p>1.$$

Se $q=0$ $\quad \lim\limits_{R\to+\infty}\int_1^R \sin x\,dx$ non esiste finito

$$\iff \lim_{R\to+\infty}\int_0^R \sin x\,dx \text{ non esiste finito}$$

Supponiamo per assurdo che esista

$$\lim_{R\to+\infty}\int_0^R \sin x\,dx = \lim_{n\to\infty}\underbrace{\int_0^{2\pi n}\sin x\,dx}_{0}$$

$$= \lim_{n\to+\infty}\int_0^{\pi+2\pi n}\sin(x)\,dx$$

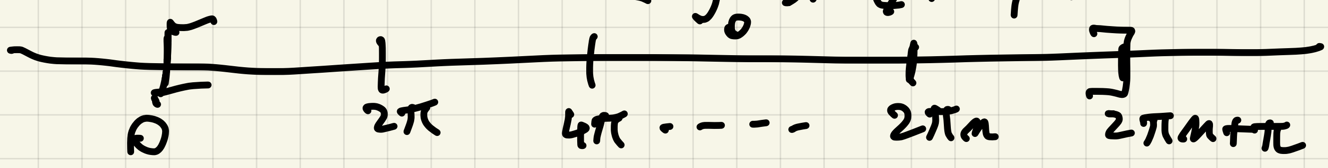
$$\int_0^{3\pi} \sin x = -\cos(x)\Big]_0^{3\pi} = -\cos(3\pi) + \cos(0) =$$

$$= -\cos(\pi) + \cos(0) = 2$$

$$\overbrace{\int_0^{\pi+2n\pi} \sin x \, dx}^{2} = \overbrace{\int_0^{2\pi n} \sin x \, dx}^{0} +$$

$$+ \int_{2\pi n}^{2\pi n+\pi} \sin(x) \, dx = \int_0^{\pi} \sin(y + 2\pi n) \, dy \qquad y = x - 2\pi n$$

$$= \int_0^{\pi} \sin(y) \, dy = -\cos\pi + \cos(0) = 2$$



$$\lim_{n \to +\infty} \int_0^{2\pi n+\pi} \sin(x) \, dx = 2$$

$$\Rightarrow \lim_{R \to +\infty} \int_0^{R} \sin(x) \, dx = 2 \Rightarrow$$

assurdo

per $p = 1$ $\quad \sin(x) \notin L[0,+\infty)$

$$\lim_{R\to+\infty} \int_1^R \frac{\sin(x)}{x^q}\,dx \qquad q = 1 - \frac{1}{p}$$

per $p > 1$ integrabile

per $0 < p < 1 \iff q < 0$

allora non è integrabile

Ho usato

$$\frac{\sin x}{x^q} \in L([1,+\infty)) \qquad \forall\, q > 0$$

$$\lim_{R\to+\infty} \int_1^R \frac{\sin x}{x^q}\,dx$$

$$\int_1^R \frac{\sin x}{x^q}\,dx = \int_1^R \frac{(-\cos x)'}{x^q}\,dx =$$

$$= -\frac{\cos x}{x^q}\Big]_1^R + \int_1^R \cos x \left(\frac{1}{x^q}\right)' dx$$

$$\int_1^R \frac{\sin x}{x^q}\, dx = \int_1^R \frac{(-\cos x)'}{x^q}\, dx =$$

$$= -\frac{\cos x}{x^q}\Big]_1^R + \int_1^R \cos x \left(\frac{1}{x^q}\right)' dx$$

$$= \cos 1 - \frac{\cos R}{R^q} - q\int_1^R \frac{\cos x}{x^{q+1}}\, dx$$

La funzione $\frac{\cos x}{x^{q+1}}$ soddisfa per $x \geq 1$

$$\alpha \leq \frac{|\cos x|}{x^{q+1}} \leq \frac{1}{x^{q+1}} \in L[1,+\infty)$$

poiché $q+1 > 1$

$\Longrightarrow$ esiste ed è in $\mathbb{R}$

$$\lim_{R\to+\infty} -q\int_1^R \frac{\cos x}{x^{q+1}}\, dx$$

$$\lim_{R\to+\infty}\int_1^R \frac{\sin x}{x^q}\, dx = \cos 1 + \lim_{R\to+\infty} -q\int_1^R \frac{\cos x}{x^{q+1}}\, dx$$

$$\Longrightarrow \frac{\sin x}{x^q} \in L[1,+\infty).$$