$V \qquad \dim V = \infty \qquad \mathcal{B} = \{e_i\}_{i \in I}$

Def. $f: V \longrightarrow K$ ponendo $f(e_i) = 1 \quad \forall i \in I$

$\{e_i^*\}_{i \in I} \qquad e_i^*: V \longrightarrow K$

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Supp. per ass. che $f = \sum_{\text{fin}} a_i e_i^*$

$\exists\, j \in I$ t.c. e_j^* non compare nella somma fin.

$f(e_j) = \sum a_i \underbrace{e_i^*(e_j)}_{=0} = 0$

ma $f(e_j) = 1$

assurdo

$V = K[X]$ $\mathcal{B} = \{1, x, x^2, \dots, x^n, \dots\}$

$f : K[x] \longrightarrow K$ t.c. $1 \to 1$

$x \to 1$

$x^2 \to 1$

$\vdots$

$x^n \to 1$

$\vdots$

$f(p(x)) = p(1)$

$p(x) = a_0 + a_1 x + \dots + a_n x^n$

$f(p(x)) = a_0 + a_1 + \dots + a_n$

f non è comb. lin. dei $(x^k)^*$

$z = a + bi$ $Re : \mathbb{C} \longrightarrow \mathbb{R}$

$z \longrightarrow a$

$Im : \mathbb{C} \longrightarrow \mathbb{R}$

$z \longrightarrow b$

sono le funz. coord. risp. alla base $(1, i)$ di $\mathbb{C}$ come $\mathbb{R}$-sp. vett.

V $\mathbb{R}$-sp. $\boxed{\mathcal{B} = (v_1, \dots, v_n)}$

$b : V \times V \longrightarrow \mathbb{R}$ forma $\boxed{\text{bilin.}}$

$$A = M_{\mathcal{B}}(b) = \left(b(v_i, v_j) \right)$$

$$b(v, w) = b\left(\sum_i \underline{x_i} v_i, \sum_j y_j v_j\right) =$$

$$= \sum_{i,j} x_i y_j \, b(v_i, v_j) = \underline{\underline{{}^tX A Y}}$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Se b è simm.
A risulta simm.

Vicev., data una matrice $n \times n$ simm.
poniamo def. un'appl. $b : V \times V \to \mathbb{R}$
ponendo $b(v, w) = {}^tX A Y$: risulta
una forma bilin. simm.

$$b(\underbrace{x_1 v_1 + \dots + x_n v_n}_{v}, \underbrace{y_1 v_1 + \dots + y_n v_n}_{w}) =$$

$$= b\left(\underline{\sum_{i=1}^n x_i v_i}, \underline{\sum_{j=1}^n y_j v_j}\right)$$

$V \times V \longrightarrow \mathbb{R}$ $\qquad$ $\underline{\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}}$

V su $\mathbb{C}$ $\qquad$ $b: V \times V \longrightarrow \mathbb{C}$

Se b è sesquilineare hermitiana

$$* \; b\left(\sum x_i v_i, \sum y_j v_j\right) =$$

$$= \boxed{\sum \overline{x_i} y_j \, b(v_i, v_j)}$$

$A = M_{\mathcal{B}}(b) = \left(b(v_i, v_j)\right)$ risulta

hermitiana: $\boxed{{}^t A = \overline{A}}$

$$A \qquad \sum \overline{x_i} y_j a_{ij}$$

$$\overline{x_j} y_i \overline{a_{ij}}$$

$$\langle z, w \rangle_3 = \overline{z_1} w_1 - i \overline{z_1} w_2 + i \overline{z_2} w_1$$

$$\begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix}$$

$$\langle z, z\rangle_3 = \overline{z_1} z_1 - i\,\overline{z_1} z_2 + i\,\overline{z_2} z_1$$

$z_1 = a_1 + b_1 i$, $\quad z_2 = a_2 + b_2 i$

$$i\,\overline{z_2} z_1 = \overline{-i\,\overline{z_1} z_2}$$

$$a + bi + a - bi = 2a$$

$z = a + bi \qquad z + \overline{z} = 2\,\mathrm{Re}\, z$

$V \supseteq W \qquad V/W$

$\pi : V \longrightarrow V/W$ lineare suriettiva: proiezione canonica

$v \longrightarrow [v]$

$V = U \oplus W$

$v = u + w$ $\boxed{\underline{\text{unica}}}$ espressione

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \\ W & & \end{array} \qquad \begin{array}{ccc} v & \longrightarrow & u \\ \downarrow & & \\ w & & \end{array}$$

lineari suriettive

$$V/W \simeq U$$

$$\begin{array}{ccc} \pi : V & \longrightarrow & V/W \\ \cup\!\vert & & \| \\ U & \xrightarrow[\pi|_U]{} & V/W \end{array}$$

$\pi|_U$ è iniettiva : $\underset{[u]}{\overset{\|}{\pi(u)}} = 0 \iff$

$V/W = V/\sim \qquad v \sim v' \iff v - v' \in W$

0 di V/W è $[0] = \{v \in V \mid v \sim 0\}$
$= W$

$\pi(u) = 0 \iff u \in W \cap U = (0)$
$\Rightarrow \ker \pi|_U = 0$

$\pi|_U$ è suriettiva :

$[v] \in V/W \qquad v = u + w \, , \, w \in W$
$\quad \| \qquad\qquad\qquad v \sim u$
$[u] = \pi|_U(u)$

$\Rightarrow \pi|_U$ è isom.

$A \quad n \times n$

A non è invertibile $\iff \exists B \neq 0$
t.c. $\boxed{AB = 0}$

$$AB = 0 \iff L(AB) = 0$$

$$L(AB) = L(A) \circ L(B)$$

$$V \xrightarrow{f} V' \xrightarrow{g} V'' \qquad g \circ f = 0$$

$$v \to f(v) \to g(f(v)) = 0$$

$$\mathrm{Im}\, f \subseteq \mathrm{Ker}\, g$$

$$L(A) \circ L(B) = 0 \iff \mathrm{Im}\, L(B) \subseteq$$

$$\subseteq \underline{\mathrm{Ker}\, L(A)}$$

$\mathrm{Im}\, L(B)$ è generata dalle colonne di B

$\mathrm{Im}\, L(B) \subseteq \mathrm{Ker}\, L(A) \iff$ le colonne di B appartengono al nucleo di $L(A)$

Questo si può ottenere con $B \neq 0$ se e solo se $\boxed{\mathrm{Ker}\, L(A) \neq (0)}$

$\iff \mathrm{rg}\, A < n \iff A$ non è invertibile

$$AB = 0 \qquad \det(AB) = 0$$
$$\det(AB) = \det(A)\det(B)$$

$$AB = 0, \quad \exists A^{-1} \Rightarrow A^{-1}(AB) = A^{-1}0 = 0$$
$$A^{-1}(AB) = B$$

$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \qquad \lambda_1 = 1, \ \lambda_2 = -3$

$\mathrm{Aut}(1) = \langle e_1 + e_3, e_2 - e_4 \rangle \qquad 2$

$\mathrm{Aut}(-3) = \langle e_1 - e_3, 2e_1 + e_2 + e_4 \rangle \qquad 2$

$m_g(1) = 2, \quad m_g(-3) = 2$

$m_g(1) \leq m_a(1) \qquad m_g(-3) \leq m_a(-3)$

$p_T(x) = (x-1)^2(x+3)^2$

$m_a = m_g$ per entrambi

Per scrivere la matrice di T risp. alla base canon.

$$B = (\underbrace{e_1 + e_3, e_2 - e_4}_{1}, \underbrace{e_1 - e_3, 2e_1 + e_2 + e_4}_{-3})$$

$$M_{\mathcal{B}}(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

$M_{\mathcal{C}}(T)$ — ——— –

$T(e_1)$

$T(e_2)$

$T(e_3)$

$T(e_4)$

$$T(e_1+e_3) = e_1+e_3$$

$$T(e_1-e_3) = -3(e_1-e_3)$$

$$T(e_1+e_3) + T(e_1-e_3) = e_1+e_3-3(e_1-e_3)$$

$$\| $$

$$T(2e_1+e_3-e_3) = T(2e_1) = 2T(e_1)$$

$$T(e_1) = \frac{1}{2}(e_1+e_3-3e_1+3e_3)$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Endomorfismi ortogonali e unitari

Endom. autoaggiunti ...

$$W \oplus W^\perp = V \qquad\qquad v = \underbrace{w}_{W} + \underbrace{w'}_{W^\perp}$$

$$W^\perp \simeq V/W \qquad\qquad V \longrightarrow W \simeq V/W^\perp$$

$$v \longrightarrow w$$

$$W \simeq V/W^\perp \qquad\qquad v \longrightarrow [v] \in V/W^\perp$$