Study of a function

1) Domain
2) Sign (find $x \in$ domain of the function such that $f(x) \geq 0$
   and in particular $f(x) = 0$)
3) Limits (at the extreme points of the domain)
4) First derivative and monotonicity
5) Second derivative and concavity
6) Graph

Lagrange's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in $]a, b[$.

Then $\exists\ \eta \in]a, b[$ such that

$$f'(\eta) = \frac{f(b) - f(a)}{b - a}$$

Consequence of Lagrange's Theorem

Let $f : I \to \mathbb{R}$ $\quad I \subset \mathbb{R}$ interval

be differentiable in I

If $f'(x) \geq 0$, $\forall x \in I$, then f is non-decreasing in I

$(\leq)$ $\qquad$ (non-increasing in I)

If $f'(x) > 0$, $\forall x \in I$, then f is increasing in I (decreasing in I)

$(<)$

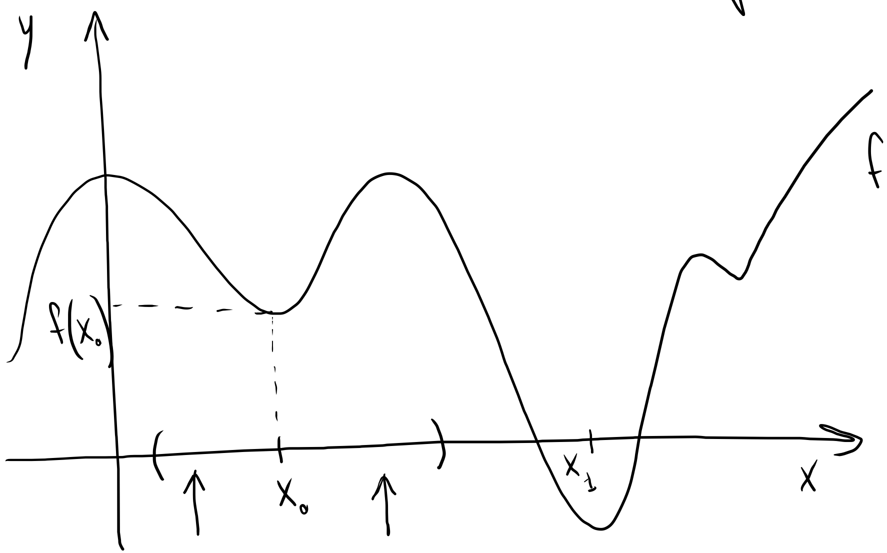
Def. of points of relative minimum / maximum

Let $f : A \longrightarrow B$ $\qquad A, B \subset \mathbb{R} \qquad x_0 \in A$

x_0 is a point of relative minimum of f if

$\exists\, U_{x_0}$ such that $\forall\, x \in U_{x_0} \cap A \setminus \{x_0\}$, $f(x) > f(x_0)$

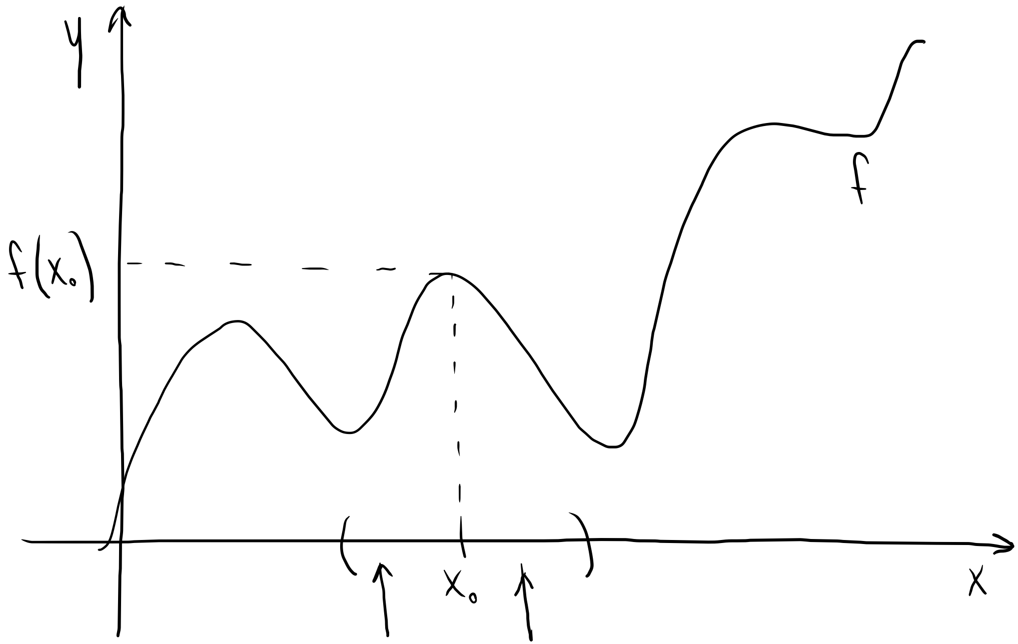
$f(x_0)$ is called a relative minimum of f



$x_0 \in A$ is a point of relative maximum of f if

$\exists\, U_{x_0}$ such that $\forall\, x \in U_{x_0} \cap A \setminus \{x_0\}$, $f(x) < f(x_0)$

$f(x_0)$ is called a relative maximum of f

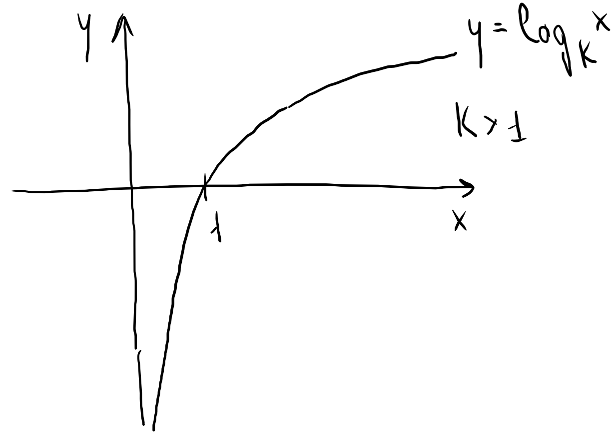


Exercises

$f(x) = x^2 \cdot \log x$ (January 27, 2009)

1) Domain $dom(f) =]0, +\infty[$

2) Sign $x^2 > 0, \forall x > 0$



$\log x \geq 0$

$\Updownarrow$

$x \geq 1$

$\log x = 0 \Leftrightarrow x = 1$

$f(x) \geq 0 \longleftrightarrow x \geq 1$

$f(x) = 0 \longleftrightarrow x = 1$

3) Limits

$$\mathrm{dom}(f) = \left]0, +\infty\right[$$

$$\lim_{x \to 0^+} x^2 \cdot \log x = 0^- \quad \text{(relevant limit)}$$

$$\lim_{x \to +\infty} x^2 \cdot \log x = +\infty$$

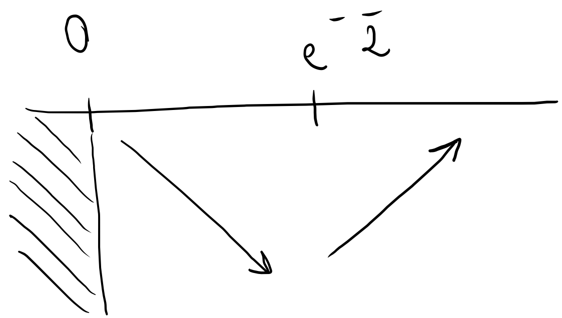
4) First derivative and monotonicity

$$f'(x) = 2x \cdot \log x + x^2 \cdot \frac{1}{x} = 2x \log x + x = x(2 \log x + 1)$$

$$f'(x) \geq 0 \longleftrightarrow 2 \log x + 1 \geq 0$$

$$\longleftrightarrow \log x \geq -\frac{1}{2}$$

$$\longleftrightarrow x \geq e^{-\frac{1}{2}}$$



$$f'(x) = 0 \longleftrightarrow x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

Because of the consequence of the Lagrange's Theorem, we conclude that f is decreasing in $\left]0, e^{-\frac{1}{2}}\right]$ and f is increasing in $\left[e^{-\frac{1}{2}}, +\infty\right[$

$x_m = e^{-\frac{1}{2}}$ point of (relative) absolute minimum

$$f\left(e^{-\frac{1}{2}}\right) = e^{-1} \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2e}$$

$$f(x) = x^2 \log x$$

absolute minimum

No other points of relative minimum or maximum of f

5) Second derivative and concavity

$$f''(x) = 2 \log x + 1 + x \cdot \frac{2}{x} = 3 + 2 \log x$$

$$f'(x) = x\,(2 \log x + 1)$$

$$f''(x) \geq 0 \longleftrightarrow 3 + 2\log x \geq 0$$

$$\longleftrightarrow \log x \geq -\frac{3}{2}$$

$$\longleftrightarrow x \geq e^{-\frac{3}{2}}$$

$$f''(x) = 0 \longleftrightarrow x = e^{-\frac{3}{2}}$$

f is concave in $\left]0, e^{-\frac{3}{2}}\right]$ and convex in $\left[e^{-\frac{3}{2}}, +\infty\right[$

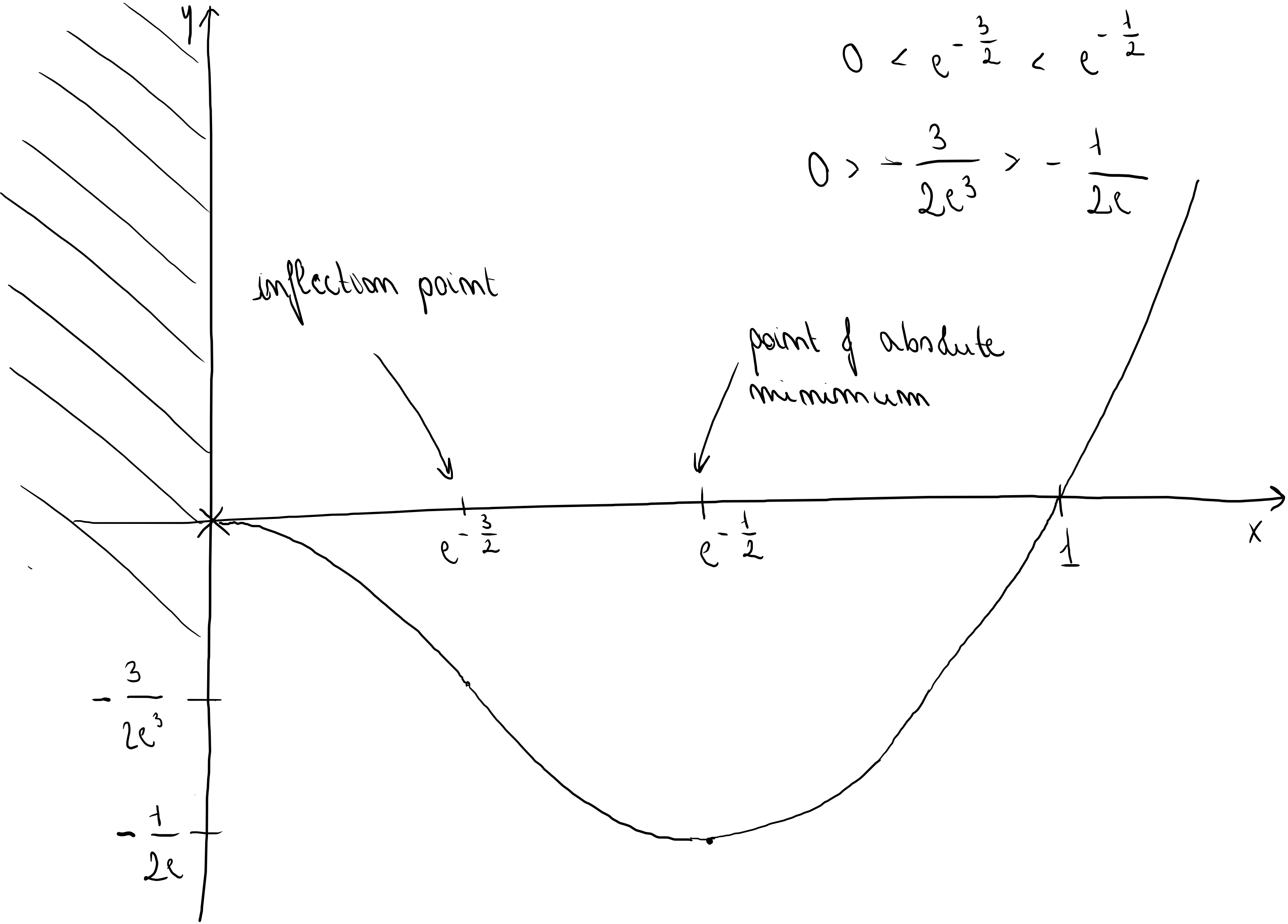
$x_i = e^{-\frac{3}{2}}$ inflection point

$$f(x_i) = e^{-3}\left(-\frac{3}{2}\right) = -\frac{3}{2e^3}$$

$$f(x) = x^2 \cdot \log x$$

$$0 < e^{-\frac{3}{2}} < e^{-\frac{1}{2}}$$

$$0 > -\frac{3}{2e^3} > -\frac{1}{2e}$$



$f(x) = \log\left(\frac{1}{x^2+1}\right)$ (June 12, 2012)

1) Domain $\frac{1}{x^2+1} > 0, \ \forall x \in \mathbb{R}$

$\mathrm{dom}(f) = \mathbb{R}$

2) Sign $\frac{1}{x^2+1} \leq 1, \ \forall x \in \mathbb{R}$

$\frac{1}{x^2+1} = 1 \longleftrightarrow x = 0$

$f(x) = \log\left(\frac{1}{x^2+1}\right) \leq 0, \ \forall x \in \mathbb{R}$

$f(x) = 0 \longleftrightarrow x = 0$

3) Symmetry f is even: $f(x) = f(-x), \ \forall x \in \mathbb{R}$

4) Limits

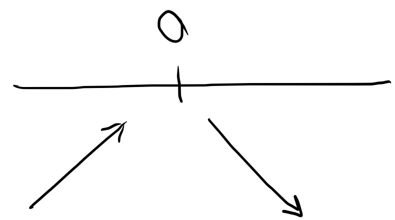
$$\lim_{x\to+\infty} \log\Bigg(\underbrace{\frac{1}{x^2+1}}_{0^+}\Bigg) = -\infty = \lim_{x\to-\infty} \log\left(\frac{1}{x^2+1}\right)$$

because f is even

5) First derivative and monotonicity

$$f(x) = g(h(x)) = g\circ h(x) \qquad h(x) = \frac{1}{x^2+1} \qquad g(t) = \log t$$

$$f'(x) = g'(h(x))\cdot h'(x)$$



$$= \cancel{(x^2+1)}\cdot\left(-\frac{1}{(x^2+1)^{\cancel{2}}}\cdot 2x\right)$$

$$= -\frac{2x}{x^2+1} \le 0 \quad \forall\, x \geq 0 \qquad f'(x) = 0 \longleftrightarrow x = 0$$

f is increasing in $]-\infty, 0]$, decreasing in $[0, +\infty[$

$\longrightarrow$ $x_M = 0$ is a point of absolute maximum

$f(0) = 0$ absolute maximum

No other relative maximum or minimum

$\inf_{\mathbb{R}} f = -\infty$

6) Second derivative and concavity

$$f'(x) = -\frac{2x}{x^2+1}$$

$$f''(x) = -\frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} = -\frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2x^2-2}{(x^2+1)^2} = 2\,\frac{x^2-1}{(x^2+1)^2}$$

$f''(x) \geq 0 \longleftrightarrow x^2 - 1 \geq 0 \longleftrightarrow x \leq -1 \vee x \geq 1$

f is convex in $]-\infty, -1]$ and in $[1, +\infty[$

concave in $[-1, 1]$

$x_1 = -1$, $x_2 = 1$ are inflection points

$f(-1) = f(1) = \log\left(\frac{1}{2}\right) \ (< 0)$

$$f(x) = \frac{\sqrt{x+1}}{x}$$

(September 10, 2013)

1) Domain

$$\begin{cases} x \neq 0 \\ x+1 \geq 0 \end{cases} \qquad \begin{cases} x \neq 0 \\ x \geq -1 \end{cases}$$

$$\mathrm{dom}(f) = [-1, 0[\cup]0, +\infty[$$

2) Sign : $f(x) \geq 0 \longleftrightarrow x > 0$

$$f(x) = 0 \longleftrightarrow \sqrt{x+1} = 0 \longleftrightarrow x = -1$$

3) Limits

$$\lim_{x \to 0^-} \frac{\sqrt{x+1}}{x} = -\infty$$

$x = 0$ vertical asymptote of f

$$\lim_{x \to 0^+} \frac{\sqrt{x+1}}{x} = +\infty$$

$$\lim_{x \to +\infty} \frac{\sqrt{x+1}}{x} = \lim_{x \to +\infty} \sqrt{\underbrace{\frac{x+1}{x^2}}_{\to 0}} = 0^+$$

$x = \sqrt{x^2}$ since $x \to +\infty$

$y = 0$ horizontal asymptote of f as $x \to +\infty$

4) First derivative and monotonicity

$$f'(x) = \frac{\frac{1}{2\sqrt{x+1}} \cdot x - \sqrt{x+1} \cdot 1}{x^2} = \frac{x - 2(x+1)}{2x^2\sqrt{x+1}}$$

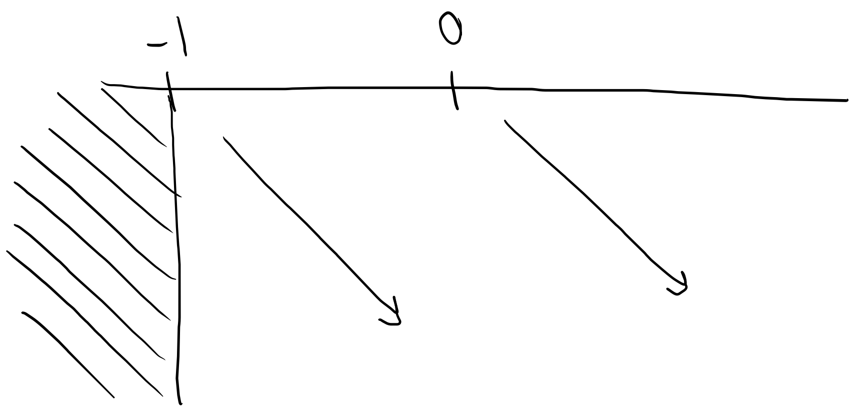
$$= -\frac{x+2}{2x^2\sqrt{x+1}}$$

$\text{dom}(f') = \text{dom}(f) \setminus \{-1\} =]-1, 0[\cup]0, +\infty[$

$\forall x \in \text{dom}(f'),\ 2x^2 \cdot \sqrt{x+1} > 0$ $\qquad f'(x) \geq 0 \longleftrightarrow x + 2 \leq 0$

$x \leq -2$

Then $f'(x) < 0,\ \forall x \in dom(f') =]-1, 0[\cup]0, +\infty[$



f is decreasing in $]-1,0[$ and in $]0,+\infty[$

$x_M = -1$ is a point of relative maximum

$f(-1) = 0$ relative maximum

No other relative minimum or maximum

$\sup f = \lim_{x \to 0^+} f(x) = +\infty$

$\nexists$ absolute minimum or maximum: indeed, $\inf f = \lim_{x \to 0^-} f(x) = -\infty$

5) Second derivative and concavity

$$f'(x) = -\frac{x+2}{2x^2 \cdot \sqrt{x+1}}$$

$$f''(x) = -\frac{2x^2 \cdot \sqrt{x+1} - (x+2) \cdot \left(4x\sqrt{x+1} + \cancel{2}x^2 \cdot \frac{1}{\cancel{2}\sqrt{x+1}}\right)}{4x^4 (x+1)}$$

$$= -\frac{2x^2(x+1) - (x+2) \cdot \left(4x \cdot (x+1) + x^2\right)}{4x^4 (x+1)^{3/2}}$$

$$= -\frac{(x+1)\left(2x^2 - 4x(x+2)\right) - (x+2)x^2}{4x^4 (x+1)^{3/2}}$$

$$= \frac{x}{4x^4 (x+1)^{3/2}} \left((x+1)(2x+8) + x(x+2)\right) = \frac{x}{4x^4 (x+1)^{3/2}} \left(2x^2 + 8x + 2x + 8 + x^2 + 2x\right)$$

$$= \frac{x}{4x^4 (x+1)^{3/2}} \left(2x^2 + 8x + 2x + 8 + x^2 + 2x\right)$$

$$= \underbrace{\frac{x}{4x^4 (x+1)^{3/2}}} \underbrace{\left(3x^2 + 12x + 8\right)} = f''(x)$$

$$3x^2 + 12x + 8 = 0$$

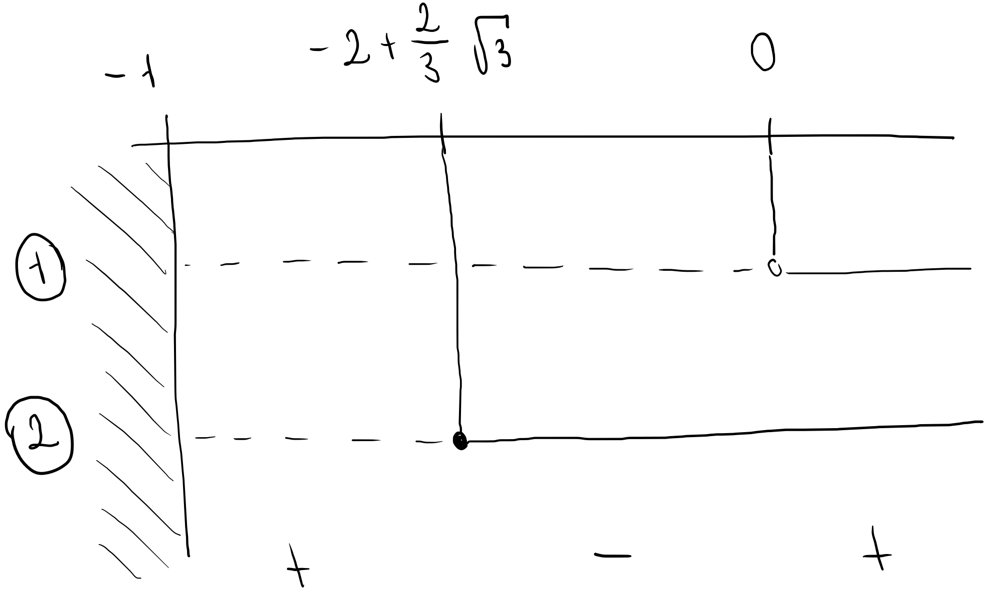
$$3x^2 + 12x + 8 \geq 0 \longleftrightarrow \cancel{x \leq -2 - \frac{2}{3}\sqrt{3}} \vee$$

$$x \geq -2 + \frac{2}{3}\sqrt{3}$$

$$\Delta = 144 - 24 \cdot 4 = 144 - 96 = 48 = 2^4 \cdot 3$$

$$x_1 = \frac{-12 - 4\sqrt{3}}{6} = -2 - \frac{2}{3}\sqrt{3} \quad (< -1)$$

$$x_2 = \frac{-12 + 4\sqrt{3}}{6} = -2 + \frac{2}{3}\sqrt{3} \quad (\in]-1, 0[)$$



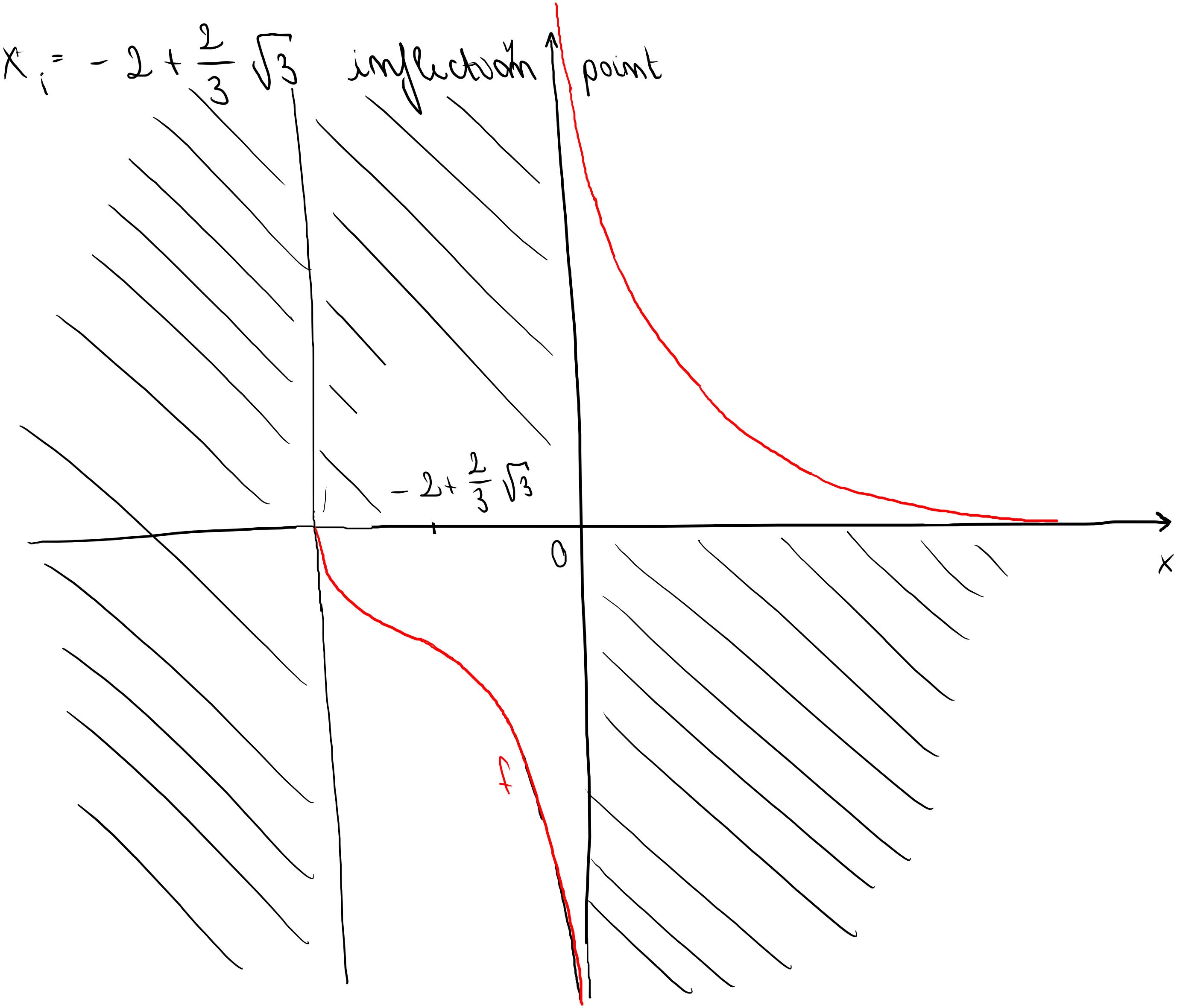
$$f''(x) = \underbrace{\frac{x}{4x^4(x+1)^{3/2}}}_{1} \cdot \underbrace{(3x^2+12x+8)}_{2}$$

$f''(x) \geq 0$ if and only if $x \in \left]-1, -2+\frac{2}{3}\sqrt{3}\right] \cup \left]0, +\infty\right[$

$f''(x) = 0$ if and only if $x = -2+\frac{2}{3}\sqrt{3}$

f is convex in $\left[-1, -2+\frac{2}{3}\sqrt{3}\right]$ and in $\left]0, +\infty\right[$

concave in $\left[-2+\frac{2}{3}\sqrt{3}, 0\right[$


$x_i = -2 + \frac{2}{3}\sqrt{3}$ inflection point
y
$-2 + \frac{2}{3}\sqrt{3}$
0
x
f