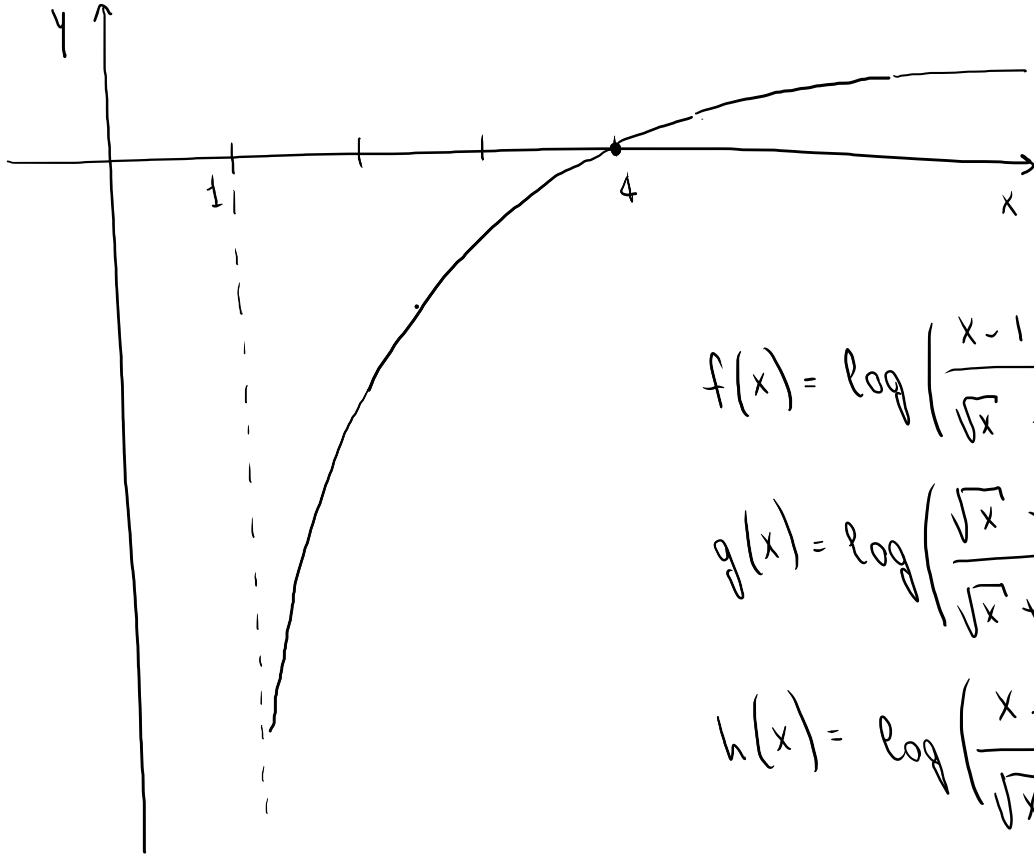


$$f(x) = \log\left(\frac{x-1}{\sqrt{x}+1}\right)$$

$$g(x) = \log\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$$

$$h(x) = \log\left(\frac{x+1}{\sqrt{x}-1}\right)$$

no one of the previous functions

$f(4) = \log\left(\frac{3}{3}\right) = \log 1 = 0$

$f(x) = \log\left(\frac{x-1}{\sqrt{x}+1}\right)$

$g(4) = \log\left(\frac{1}{3}\right) < 0$

$g(x) = \log\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$

g cannot be the correct law

$h(4) = \log\left(\frac{5}{1}\right) = \log 5 > 0$

$h(x) = \log\left(\frac{x+1}{\sqrt{x}-1}\right)$

h cannot be the correct law

1) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \log\left(\frac{x-1}{\sqrt{x}+1}\right) = -\infty$

$\frac{x-1}{\sqrt{x}+1} \to 0^+$

$x = 1$ vertical asymptote of f

0) dom(f) = ?

$\text{dom}(f) =]1, +\infty[$

$$\begin{cases} x \geq 0 \\ \frac{x-1}{\sqrt{x}+1} > 0 \end{cases} \quad \begin{cases} x \geq 0 \\ x > 1 \end{cases} \quad x > 1$$

2) $f'(x) = \dfrac{\cancel{\sqrt{x}+1}}{x-1} \cdot \dfrac{\sqrt{x}+1-(x-1)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^{\cancel{2}}}$

$$= \frac{(\sqrt{x}+1)2\sqrt{x}-(x-1)}{(x-1)(\sqrt{x}+1)2\sqrt{x}}$$

$$= \frac{2x+2\sqrt{x}-x+1}{(x-1)(\sqrt{x}+1)2\sqrt{x}}$$

$$= \frac{x+2\sqrt{x}+1}{(x-1)(\sqrt{x}+1)2\sqrt{x}} > 0$$

$\forall x > 1$

$f(x) = \log\left(\dfrac{x-1}{\sqrt{x}+1}\right)$

$x \xmapsto{f_1} \underbrace{\dfrac{x-1}{\sqrt{x}+1}}_{= t} \xmapsto{f_2} \log t$

$f_1(x) = \dfrac{x-1}{\sqrt{x}+1}$

$f_2(t) = \log t$

$f(x) = f_2 \circ f_1(x)$

$f'(x) = f_2'(f_1(x)) \cdot f_1'(x)$

$\forall x > 1$, $f'(x)$ is well-defined

and $x + 2\sqrt{x} + 1 > 0$

and $(x-1)(\sqrt{x}+1)2\sqrt{x} > 0$

$$f(x) = f_2 \circ f_1(x)$$

$$f_2(t) = \log t \qquad f_2'(t) = \frac{1}{t}$$

$$f_1(x) = \frac{x-1}{\sqrt{x}+1} \qquad f_1'(x) = \frac{\sqrt{x}+1 - (x-1)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

$$f'(x) = \underbrace{f_2'(f_1(x))} \cdot \underbrace{f_1'(x)}$$

$$f_2'\left(\frac{x-1}{\sqrt{x}+1}\right) = \frac{\sqrt{x}+1}{x-1}$$

$$= \frac{\sqrt{x}+1}{x-1} \cdot \frac{\sqrt{x}+1-(x-1)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

$$= \frac{\sqrt{x}+1}{x-1} \cdot \frac{(\sqrt{x}+1)\,2\sqrt{x} - (x-1)}{(\sqrt{x}+1)^2\, 2\sqrt{x}} = \frac{\sqrt{x}+1}{x-1} \cdot \frac{2x + 2\sqrt{x} - x + 1}{(\sqrt{x}+1)^2\, 2\sqrt{x}}$$

$$= \frac{\cancel{\sqrt{x}+1}}{x-1} \cdot \frac{x+2\sqrt{x}+1}{(\sqrt{x}+1)^2\, 2\sqrt{x}}$$

$$= \frac{x+2\sqrt{x}+1}{\underbrace{(x-1)}_{>0}\underbrace{(\sqrt{x}+1)}_{>0}\underbrace{2\sqrt{x}}_{>0}} = f'(x) > 0, \quad \forall x \in]1,+\infty[$$

$$\underbrace{x+2\sqrt{x}}_{>0}+1 > 0$$

As a consequence of Lagrange's theorem, f is increasing in $]1, +\infty[$

$f(4) = 0$

Then, $\forall\, x \in]1,4[,\ f(x) < 0$

$\forall\, x \in]4,+\infty[,\ f(x) > 0$

$$\lim_{x \to +\infty} \log\left(\underbrace{\frac{x-1}{\sqrt{x}+1}}_{\to +\infty}\right) = +\infty$$

Correct answer $f(x) = \log\left(\frac{x-1}{\sqrt{x}+1}\right)$

June 28, 2016

1) Domain

$f(x) = \log(1 - \log x)$ $\qquad$ $\mathrm{dom}(f) =]0, e[$

$\begin{cases} x > 0 \\ 1 - \log x > 0 \end{cases}$ $\qquad$ $\begin{cases} x > 0 \\ \log x < 1 \end{cases}$ $\qquad$ $0 < x < e$

$\log x < 1$ if and only if $x < e$

2) Find the inverse of a function given

$f(x) = \arcsin(1 - \sqrt{x})$ $\qquad$ $\begin{cases} x \geq 0 \\ -1 \leq 1 - \sqrt{x} \leq 1 \end{cases}$

domain of f

$$\begin{cases} x \geq 0 \\ 1 - \sqrt{x} \geq -1 \\ \cancel{1 - \sqrt{x} \leq 1} \end{cases}$$ is always satisfied because $\sqrt{x} \geq 0, \ \forall \ x \geq 0$

$$\begin{cases} x \geq 0 \\ \sqrt{x} \leq 2 \end{cases} \qquad 0 \leq x \leq 4 \qquad \text{dom}(f) = [0, 4]$$

$f(x) = \arcsin\left(1 - \sqrt{x}\right)$

$f(x) = f_2 \circ f_1(x)$

$$x \overset{f_1}{\longmapsto} \underbrace{1 - \sqrt{x}}_{= t} \overset{f_2}{\longmapsto} \arcsin t$$

f_1 is decreasing, f_2 is increasing $\longrightarrow$ $f = f_2 \circ f_1$ is decreasing

$f_1(x) = 1 - \sqrt{x}$

$f_2(t) = \arcsin t$

Then f is injective.

Image set of f

$$f(0) = \arcsin 1 = \frac{\pi}{2}$$

$$f(4) = \arcsin(1-2) = \arcsin(-1) = -\frac{\pi}{2}$$

$$\operatorname{Im}(f) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$f : [0,4] \longrightarrow \underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{= \operatorname{Im}(f)}$ is injective and surjective

then f is invertible

$$f^{-1} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [0,4]$$

$$y = \arcsin\left(1 - \sqrt{x}\right)$$

$$\sin y = 1 - \sqrt{x}$$

$$\sin y = 1 - \sqrt{x}$$

$$\left(\sqrt{x}\right)^2 = \left(1 - \sin y\right)^2$$

$$\boxed{\begin{gathered} x = \left(1 - \sin y\right)^2 = f^{-1}(y) \\ f^{-1} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [0, 4] \end{gathered}}$$

3) Limit $\lim_{x \to 0} \dfrac{\sin(\sin x)}{\log(x+1)}$ $\left(I.F. \ \dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\sin(\sin x)}{\log(x+1)} \cdot \frac{\sin x}{\sin x} \cdot \frac{x}{x}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$= \lim_{x \to 0} \underbrace{\frac{\sin(\sin x)}{\sin x}}_{\downarrow \, 1} \cdot \underbrace{\frac{x}{\log(x+1)}}_{\downarrow \, 1} \cdot \underbrace{\frac{\sin x}{x}}_{\downarrow \, 1} = 1$$

$$\lim_{x \to 0} \frac{\log(x+1)}{x} = 1$$

$$\lim_{x \to 0} \frac{\sin(\sin x)}{\sin x} = \lim_{t \to 0} \frac{\sin t}{t} = 1$$

$t = \sin x$

$x \to 0 \quad t \to 0$

4) Study of a function

$$f(x) = e^{-\frac{1}{x-1}}$$

$$\text{dom}(f) = \mathbb{R} \setminus \{1\} = \left]-\infty, 1\right[\cup \left]1, +\infty\right[$$

Sign $f(x) > 0, \ \forall x \in \text{dom}(f)$

Limits $\lim_{x \to -\infty} e^{-\frac{1}{x-1}} = 1 = \lim_{x \to +\infty} e^{-\frac{1}{x-1}}$

$y = 1$ horizontal asymptote

$\lim_{x \to 1^-} e^{\underbrace{-\frac{1}{x-1}}_{+\infty}} = +\infty$ $\quad$ $x = 1$ vertical asymptote

$\lim_{x \to 1^+} e^{\underbrace{-\frac{1}{x-1}}_{-\infty}} = 0^+$

First derivative

$$f(x) = e^{-\frac{1}{x-1}}$$

$$f(x) = f_2 \circ f_1(x)$$

$$x \overset{f_1}{\longmapsto} \underbrace{-\frac{1}{x-1}}_{= t} \overset{f_2}{\longmapsto} e^{t}$$

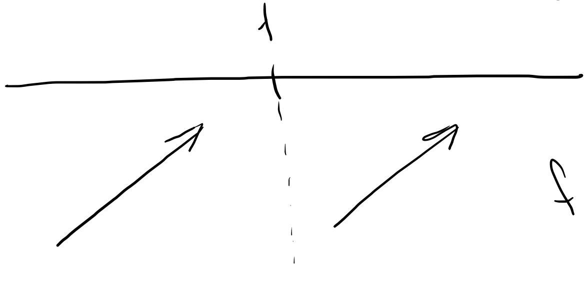
$$f_1(x) = -\frac{1}{x-1} \qquad f_2(t) = e^{t}$$

$$f_1'(x) = \frac{1}{(x-1)^2} \qquad f_2'(t) = e^{t}$$

$$f_2'(f_1(x)) = e^{-\frac{1}{x-1}}$$

$$f'(x) = f_2'(f_1(x)) \cdot f_1'(x) = \underbrace{e^{-\frac{1}{x-1}}}_{>0} \cdot \underbrace{\frac{1}{(x-1)^2}}_{>0} > 0$$

$$\forall x \in dom(f') = dom(f) =]-\infty, 1[\cup]1, +\infty[$$



f is increasing in $]-\infty, 1[$ and in $]1, +\infty[$

No relative minimum or maximum

$$\sup_{\mathrm{dom}(f)} f = +\infty$$

$$\inf_{\mathrm{dom}(f)} f = 0$$

Second derivative

$$f'(x) = e^{-\frac{1}{x-1}} \cdot \frac{1}{(x-1)^2}$$

$$f''(x) = e^{-\frac{1}{x-1}} \cdot \frac{1}{(x-1)^2} \cdot \frac{1}{(x-1)^2} + e^{-\frac{1}{x-1}} \cdot \left(-\frac{2}{(x-1)^3}\right)$$

$$= e^{-\frac{1}{x-1}} \cdot \frac{1}{(x-1)^4} + e^{-\frac{1}{x-1}} \cdot \left(-\frac{2}{(x-1)^3}\right)$$

$$= e^{-\frac{1}{x-1}} \cdot \frac{1}{(x-1)^4}\left[1 - 2(x-1)\right]$$

$$= \underbrace{e^{-\frac{1}{x-1}}}_{\substack{\vee \\ 0}} \cdot \underbrace{\frac{1}{(x-1)^4}}_{\substack{\vee \\ 0}} \cdot \underbrace{(3-2x)} = f''(x)$$

$f''(x) \geq 0$ if and only if $3 - 2x \geq 0$

$$\Updownarrow$$

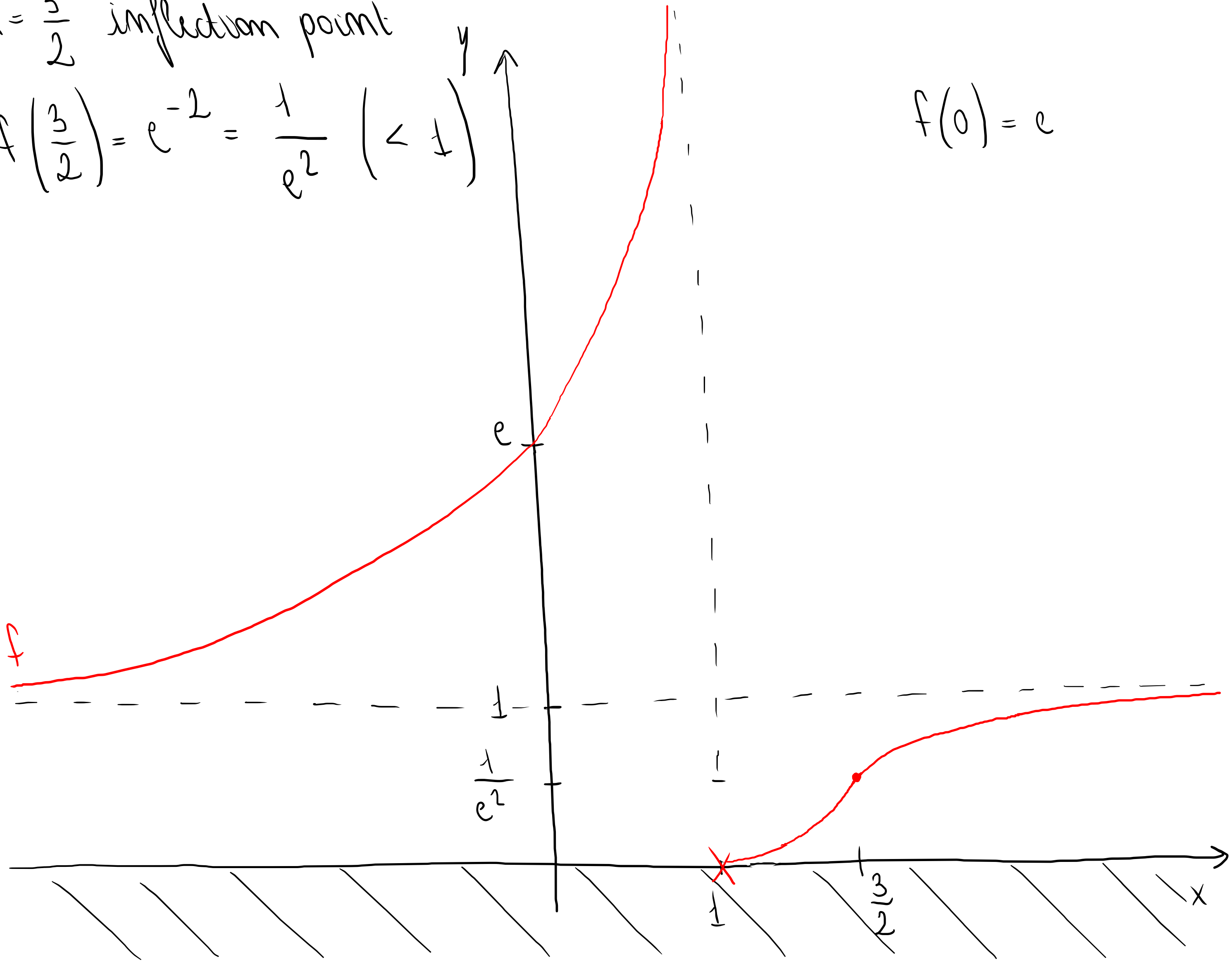
$$x \leq \frac{3}{2}$$

f is convex in $]-\infty, 1[$ and $]1, \frac{3}{2}[$, concave in $[\frac{3}{2}, +\infty[$

$x = \frac{3}{2}$ inflection point

$f\left(\frac{3}{2}\right) = e^{-2} = \frac{1}{e^2} \quad \left(< 1\right)$

$f(0) = e$



5) Integral

Definition of primitive of a function

$f : A \to \mathbb{R} \quad (A \subset \mathbb{R})$

F is a primitive of f if $F'(x) = f(x) \quad \forall x \in A$

Definition of indefinite integral of a function

$$\int f(x)\,dx = \{\text{primitives of } f\}$$

Consequence of Lagrange Theorem

If $f : I \to \mathbb{R}$ $\quad I \subset \mathbb{R}$ interval and F is a primitive of f,

then $\int f(x)\,dx = \{F(x) + c : c \in \mathbb{R}\} = F(x) + c$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx =$$

$$= \frac{2}{2}\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx =$$

$$= 2\int \cos\underbrace{\sqrt{x}}_{=t}\;\underbrace{\frac{dx}{2\sqrt{x}}}_{=dt}$$

$$= 2\int \cos t\,dt$$

$$= 2\cdot\sin t + c$$

$$= 2\cdot\sin\sqrt{x} + c$$

$$f(x) = \frac{\cos\sqrt{x}}{\sqrt{x}}$$

$$\mathrm{dom}(f) = \left]0, +\infty\right[$$

interval

$$t = \sqrt{x}$$

$$dt = \frac{1}{2\sqrt{x}}\,dx$$

$$\frac{d}{dx}\left(2\cdot\sin\sqrt{x} + c\right) = \frac{d}{dx}\left(2\cdot\sin\sqrt{x}\right) = \cancel{2}\cdot\cos\sqrt{x}\cdot\frac{1}{\cancel{2}\sqrt{x}} = \frac{\cos\sqrt{x}}{\sqrt{x}} = f(x)$$

$$\lim_{x \to 0} \frac{x - \log(1+x)}{\sin^2 x} \overset{H}{=} \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{1}{2} \qquad \left(\text{I.F. } \frac{0}{0}\right)$$

$$g(x) = \sin^2 x$$

$$g'(x) = 2 \cdot \sin x \cdot \cos x$$

$$\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\} = U_0 \setminus \{0\}$$

g' does not vanish at any $x \in U_0 \setminus \{0\}$

$$f(x) = x - \log(1+x)$$

$$f'(x) = 1 - \frac{1}{1+x}$$

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} =$$

$$\lim_{x \to 0} \frac{1 - \frac{1}{1+x}}{2 \sin x \cos x} =$$

$$\lim_{x \to 0} \frac{\frac{x}{1+x}}{2 \sin x \cos x} = \lim_{x \to 0} \frac{x}{(1+x)\, 2 \sin x \cdot \cos x} = \lim_{x \to 0} \underbrace{\frac{x}{\sin x}}_{\downarrow \, 1} \cdot \underbrace{\frac{1}{2(1+x)\cos x}}_{\downarrow \, \frac{1}{2}} = \frac{1}{2}$$