Mathematics Class

November 23, 2020

Exercise 1. Let us consider the following function:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} -x+1 & \text{if } x \leq 1, \\ \sqrt[3]{x-1} & \text{if } x > 1. \end{cases}$$

Determine

i) the sign of f;

ii) the image of f, $f(\mathbb{R})$;

iii) $\lim_{x\to-\infty} f(x)$, $f(0)$, $f(1)$, $f(9)$, $\lim_{x\to+\infty} f(x)$.

Moreover establish if the function f is continuous in its domain and draw its graph.

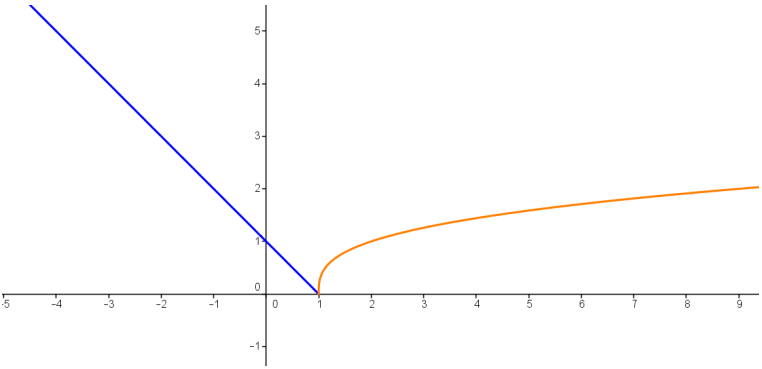
Solutions

i) $f(x) \geq 0$ for all $x \in \mathbb{R}$, with $f(x) = 0$ if and only if $x = 1$;

ii) $f(\mathbb{R}) = [0, +\infty[$;

iii) $\lim_{x\to-\infty} f(x) = +\infty$, $f(0) = 1$, $f(1) = 0$, $f(9) = 2$, $\lim_{x\to+\infty} f(x) = +\infty$.

The function f is continuous in $\mathbb{R}$. Its graph is represented here:



Exercise 2. Let us consider the following function:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{x+2} & \text{if } x < -2, \\ (x+2)(x-4) & \text{if } -2 \leq x < 4, \\ \frac{1}{2}(4-x) & \text{if } 4 \leq x. \end{cases}$$

Determine

i) the sign of f;

ii) the image of f, $f(\mathbb{R})$;

iii) $\lim_{x\to-\infty} f(x)$, $f(-3)$, $\lim_{x\to(-2)^-} f(x)$, $f(-2)$, $\lim_{x\to 4} f(x)$, $f(6)$, $\lim_{x\to+\infty} f(x)$.

Moreover say whether the function f is continuous in its domain and draw its graph.

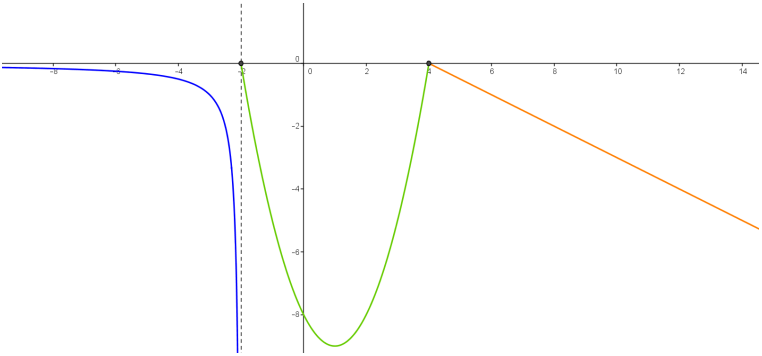
Solutions

i) $f(x) \leq 0$ for all $x \in \mathbb{R}$, with $f(x) = 0$ if and only if $x \in \{-2, 4\}$;

ii) $f(\mathbb{R}) =]-\infty, 0]$;

iii) $\lim_{x\to-\infty} f(x) = 0$, $f(-3) = -1$, $\lim_{x\to(-2)^-} f(x) = -\infty$, $f(-2) = 0$, $\lim_{x\to 4} f(x) = 0$, $f(6) = -1$, $\lim_{x\to+\infty} f(x) = -\infty$.

The function f is continuous in $]-\infty, -2[\cup]-2, +\infty[$, but not at $x = -2$, since $\lim_{x\to(-2)^-} f(x) = -\infty$. The graph of f is represented below:



Exercise 3. Compute the following limits, whenever it is possible:

1) $\lim_{x\to-2}(5x+7)$,
2) $\lim_{x\to 0}(x^2+3)$,
3) $\lim_{x\to-1}(x^3-3)$,
4) $\lim_{x\to 2}\frac{3}{x+1}$,
5) $\lim_{x\to 3}\frac{x^2-5x+6}{x-3}$,
6) $\lim_{x\to+\infty}(x^3+2)$,
7) $\lim_{x\to 2}\sqrt{3x+3}$,
8) $\lim_{x\to\pi}\cos(3x)$,
9) $\lim_{x\to+\infty} x\cos\left(\frac{1}{x}\right)$,
10) $\lim_{x\to 0}\frac{7}{x}$,
11) $\lim_{x\to 0}\frac{7}{x^2}$,
12) $\lim_{x\to+\infty}\frac{2}{x-6}$,
13) $\lim_{x\to+\infty}(x^5-x^3+2x-2018)$,
14) $\lim_{x\to\sqrt{5}}\frac{1}{x^2-5}$,
15) $\lim_{x\to 5}\frac{1-x}{(x-5)^2}$,
16) $\lim_{x\to-2}\frac{x^2-4}{x+2}$,
17) $\lim_{x\to-\infty}\sin\left(\frac{x}{2}\right)$,
18) $\lim_{x\to+\infty}\frac{\arctan x}{3}$,
19) $\lim_{x\to 0}\sqrt{1+\frac{1}{x^2}}$,
20) $\lim_{x\to 0}\sqrt[3]{1+\frac{1}{x}}$,
21) $\lim_{x\to+\infty}\frac{\cos^2 x}{x}$.

Solutions

1) -3,
2) 3,
3) -4,
4) 1,
5) 1,
6) $+\infty$,
7) 3,
8) -1,
9) $+\infty$,
10) it does not exist,
11) $+\infty$,
12) 0,
13) $+\infty$,
14) it does not exist,
15) $-\infty$,
16) -4,
17) it does not exist,
18) $\frac{\pi}{6}$,
19) $+\infty$,
20) it does not exist,
21) 0.

Exercise 4. Compute the following limits of sequences, whenever it is possible:

1) $\lim_{n\to+\infty} 5n(n-1)(n+\pi)$,
2) $\lim_{n\to+\infty} \frac{4n}{2n+5}$,
3) $\lim_{n\to+\infty} \frac{4-n^2}{n-2}$,
4) $\lim_{n\to+\infty} \frac{9n^2+2}{6-n+n^2}$,
5) $\lim_{n\to+\infty} \frac{2-n^3}{\sqrt[3]{2}-n}$,
6) $\lim_{n\to+\infty} \frac{n-2-3n^2}{4-5n^2+6n^3}$,
7) $\lim_{n\to+\infty} \frac{\sin(n\pi)}{n}$,
8) $\lim_{n\to+\infty} \cos(n^2\pi)$,
9) $\lim_{n\to+\infty} (-1)^n \frac{n^2+2}{n-2}$,
10) $\lim_{n\to+\infty} (-1)^n \frac{n+\pi}{3\pi-n}$,
11) $\lim_{n\to+\infty} \left(\sqrt{n-5}-\sqrt{n+3}\right)$,
12) $\lim_{n\to+\infty} (-1)^n \frac{\arctan n}{n}$,
13) $\lim_{n\to+\infty} \left(n-3\sqrt{n}\right)$,
14) $\lim_{n\to+\infty} (-1)^n \frac{7n}{n^2+5}$,
15) $\lim_{n\to+\infty} (-1)^n \Big(\cos^2(n\pi)-1\Big)$.

Solutions

1) $+\infty$,
2) 2,
3) $-\infty$,
4) 9,
5) $+\infty$,
6) 0,
7) 0,
8) it does not exist,
9) it does not exist,
10) it does not exist,
11) 0,
12) 0,
13) $+\infty$,
14) 0,
15) 0.

Exercise 5. The following limits are indeterminate forms. Compute them by applying suitable relevant limits, when they are useful.

1) $\lim_{x\to-\infty} \frac{2x-1}{x+3}$,
2) $\lim_{x\to+\infty} \frac{3x}{x-\sqrt{x}}$,
3) $\lim_{x\to-\infty} \frac{x^5+2x^3+1}{x^2+7x+4}$,
4) $\lim_{x\to0} \frac{\sin(\pi x)}{3x}$,
5) $\lim_{x\to\frac{\pi}{4}} \frac{\cos x-\sin x}{\cos^2 x-\sin^2 x}$,
6) $\lim_{x\to+\infty} \frac{x-x^2}{x^3-x^4}$,
7) $\lim_{x\to+\infty} \frac{1-2x^3}{x+2x^2+3x^3}$,
8) $\lim_{x\to+\infty} \frac{3-x-x^3}{1-2x^2}$,
9) $\lim_{x\to0} \frac{\sin(x^2+x)}{x}$,
10) $\lim_{x\to0^+} \frac{1-\cos\sqrt{x}}{x}$,
11) $\lim_{x\to1} \frac{\sin(\sqrt{x}-1)}{x-1}$,
12) $\lim_{x\to0} \frac{x\sin x}{\cos x-1}$,
13) $\lim_{x\to-\infty} \frac{(1-x^2)(1+x^2)}{x-x^4}$,
14) $\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x-\cos x}$,
15) $\lim_{x\to-1} \frac{x^2-2x-3}{\arcsin(x+1)}$.

Solutions

1) 2,
2) 3,
3) $-\infty$,
4) $\frac{\pi}{3}$,
5) $\frac{\sqrt{2}}{2}$,
6) 0,
7) $-\frac{2}{3}$,
8) $+\infty$,
9) 1,
10) $\frac{1}{2}$,
11) $\frac{1}{2}$,
12) -2,
13) 1,
14) $-\sqrt{2}$,
15) -4.